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SOLUTIONS MANUAL

MECHANICAL VIBRATIONS

FIFTH EDITION

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University of Miami

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To Lord Sri Venkateswara

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The MATLAB programs given in the book and the Solutions Manual, answers to problems, and answers to review questions can be found at the web site of the book: http://www.prenhall.com/rao.

The programs and techniques presented in the book, solutions manual and the web site are intended for use by students in learning the material. Although the material has been tested, no warranty is implied as to their accuracy. Solutions to few problems are missing in the Solutions Manual at this time; they will be added in few weeks.

I would appreciate receiving any errors found in the book, solutions manual or the web site of the book. The errors detected will be posted at the web site of the book.

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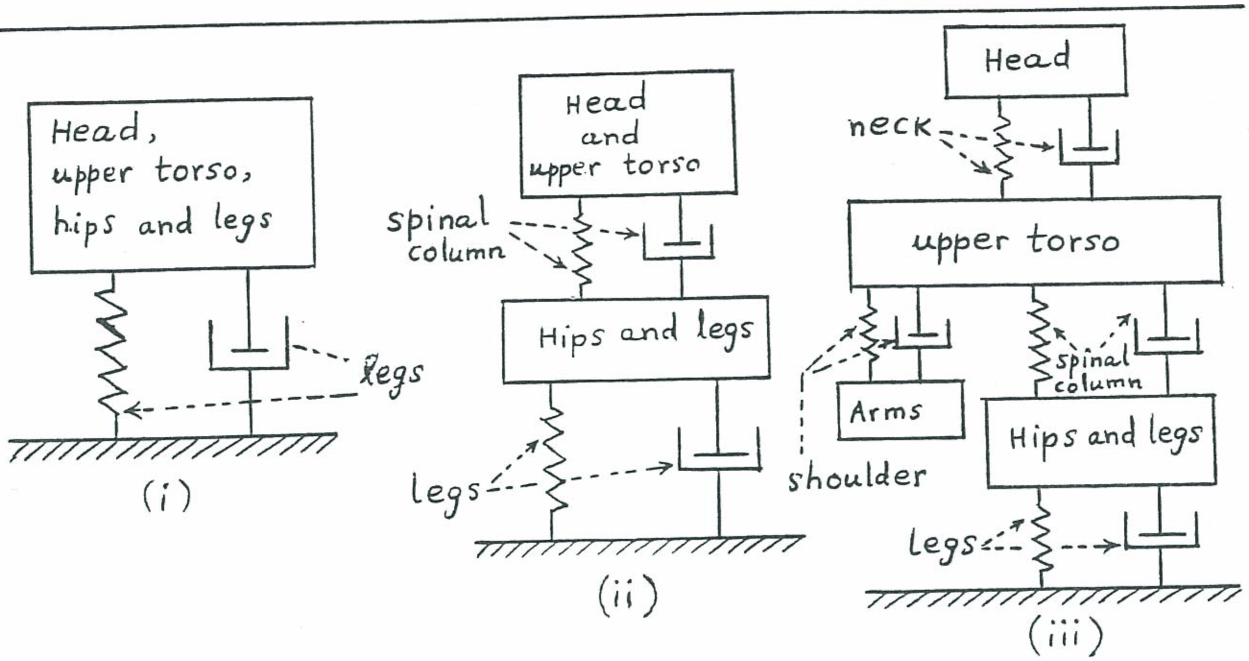
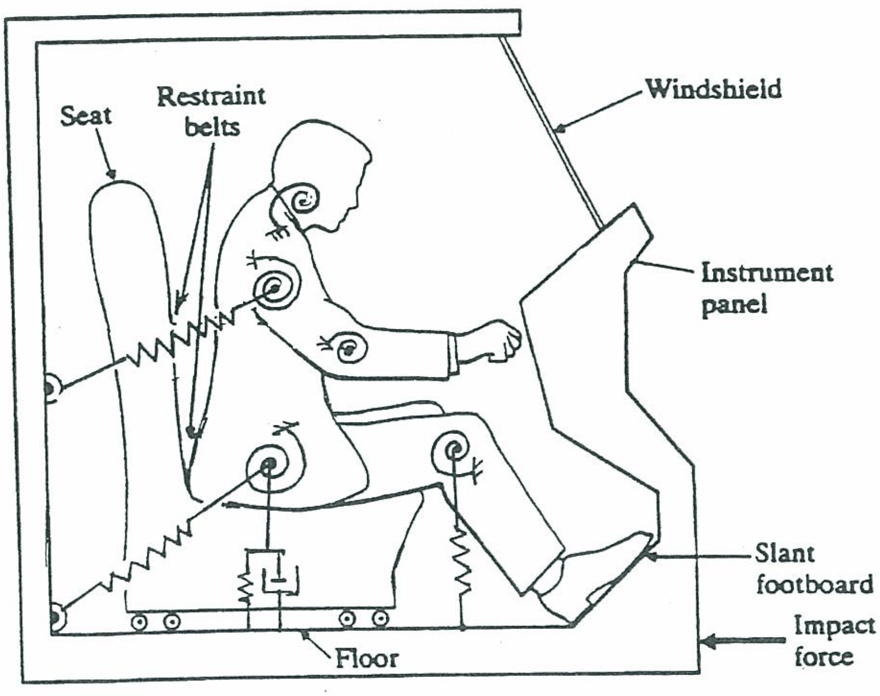
Chapter 1

Fundamentals of Vibration

1.1



1.2



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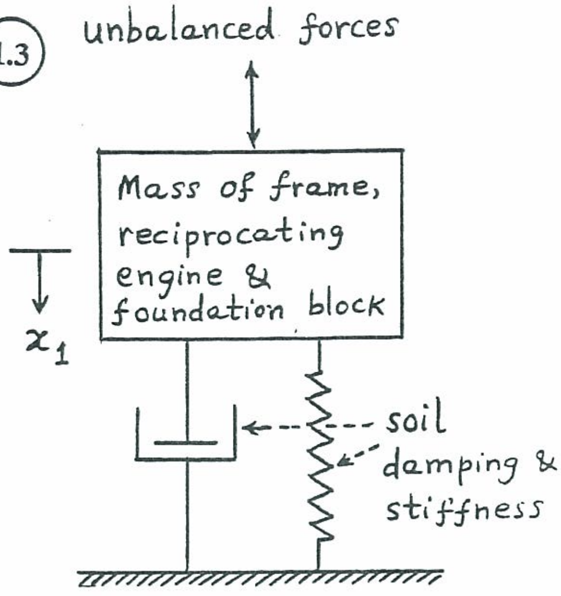
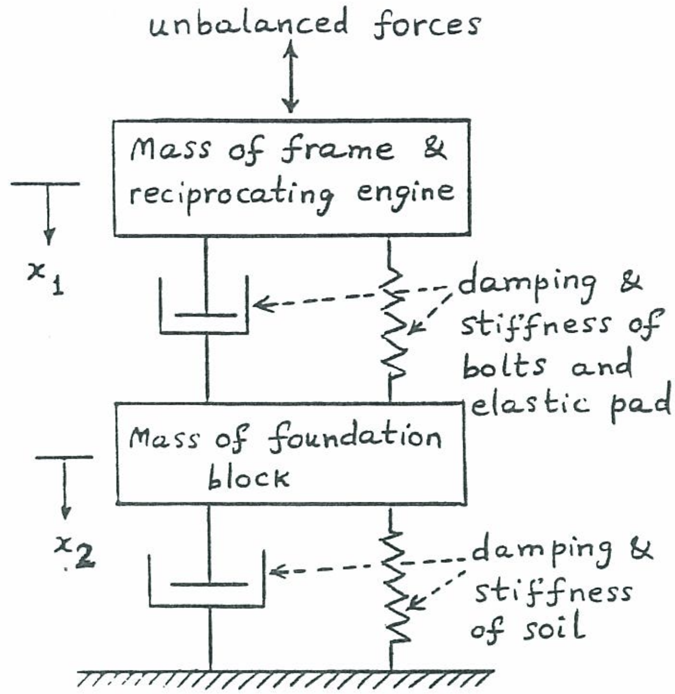
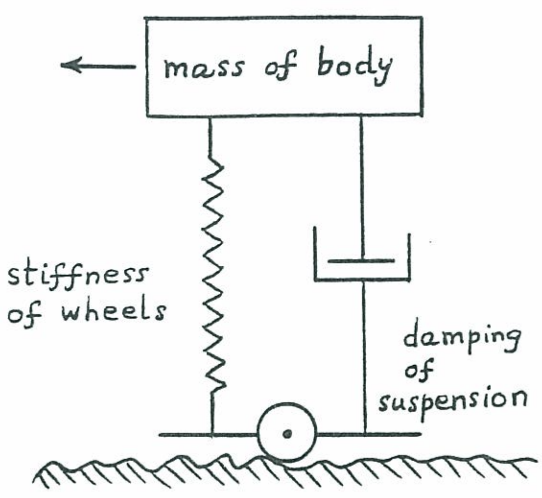


(a) one degree of freedom model

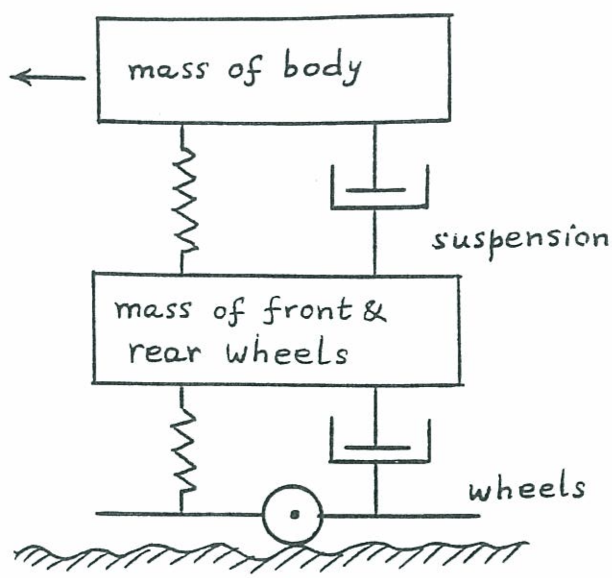


(b) Two degree of freedom model

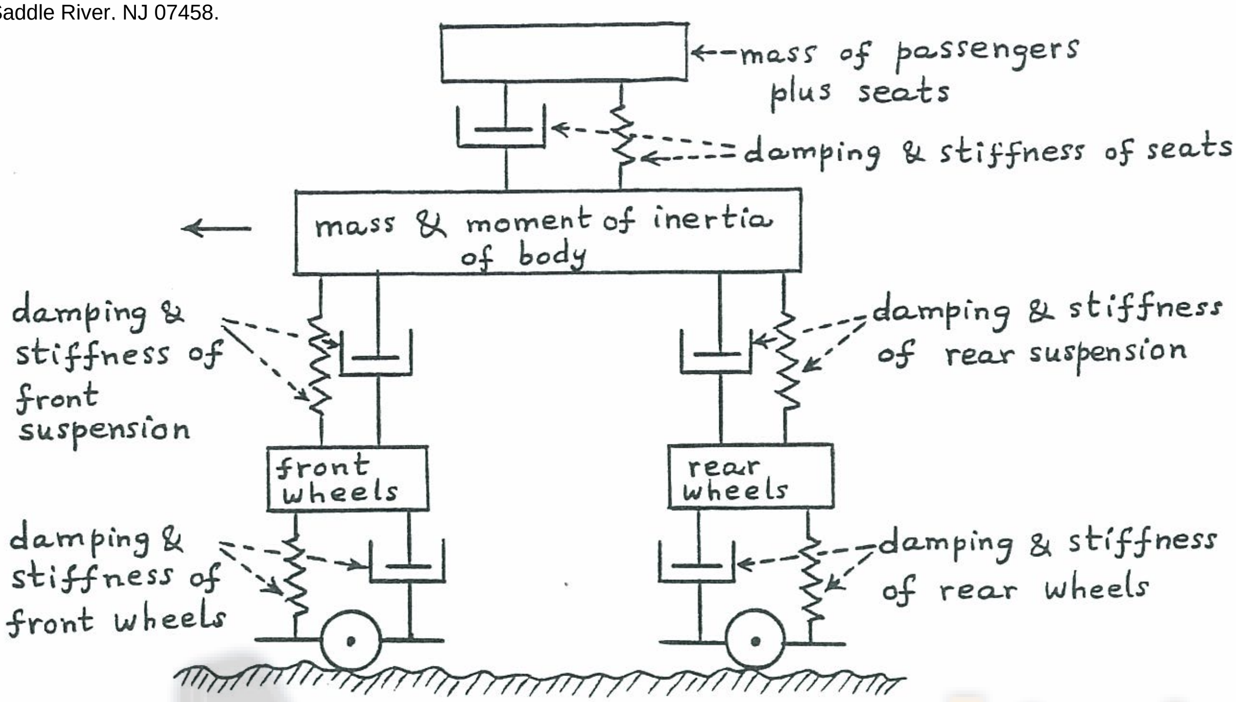
1.4



(i)

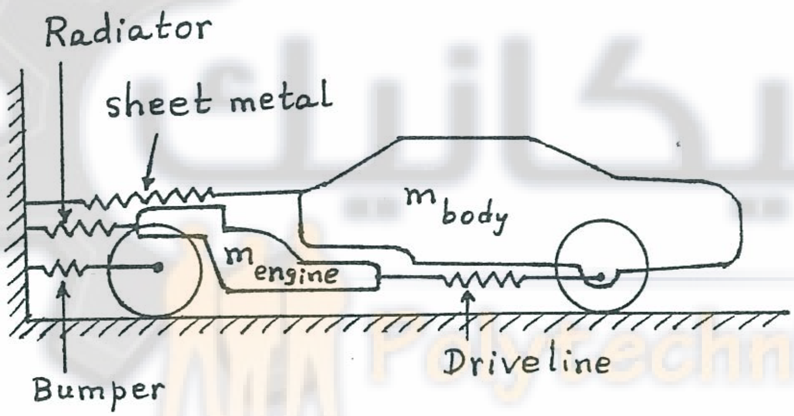


(ii)

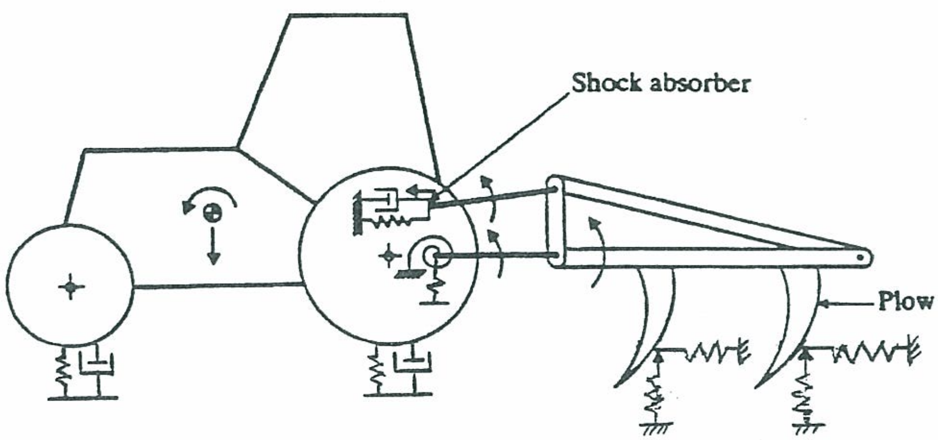

mass of passengers plus seats
damping & stiffness of seats
mass & moment of inertia of body
damping & stiffness of front suspension
damping & stiffness of rear suspension
front wheels
rear wheels
damping & stiffness of front wheels
damping & stiffness of rear wheels

(iii)

1.5

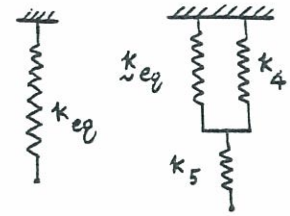

Radiator
sheet metal
m_{body}
m_{engine}
Bumper
Driveline

1.6


Shock absorber
Plow

1.7

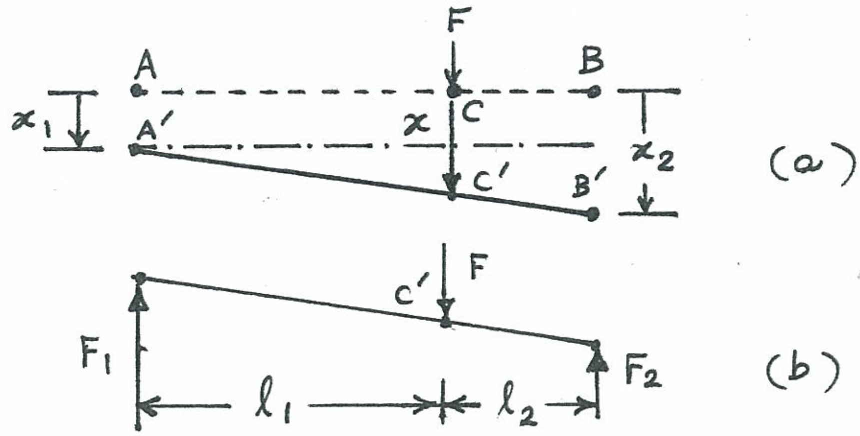
$$\frac{1}{\tilde{k}_{eq}} = \frac{1}{2k_1} + \frac{1}{k_2} + \frac{1}{2k_3} \quad ; \quad \tilde{k}_{eq} = \left(\frac{2k_1k_2k_3}{k_2k_3 + 2k_1k_3 + k_1k_2}\right)$$



$$\frac{1}{k_{eq}} = \frac{1}{\tilde{k}_{eq} + k_4} + \frac{1}{k_5}$$

$$k_{eq} = \frac{k_5(\tilde{k}_{eq} + k_4)}{k_5 + k_4 + \tilde{k}_{eq}} = \frac{k_2k_3k_4k_5 + 2k_1k_3k_4k_5 + k_1k_2k_4k_5 + 2k_1k_2k_3k_5}{k_2k_3k_4 + k_2k_3k_5 + 2k_1k_3k_4 + 2k_1k_3k_5 + k_1k_2k_4 + k_1k_2k_5 + 2k_1k_2k_3}$$

1.8



From Fig. (a), $x = x_1 + \frac{l_1}{l_1+l_2}(x_2 - x_1)$

$$= \frac{l_2}{l_1+l_2} x_1 + \frac{l_1}{l_1+l_2} x_2 \quad (1)$$

Vertical force equilibrium from Fig. (b):

$$F = F_1 + F_2 \quad (2)$$

Moment equilibrium about C' (Fig. (b)):

$$F_2 l_2 = F_1 l_1 \quad (3)$$

Solution of Eqs. (2) and (3):

$$F_1 = \frac{F l_2}{l_1 + l_2}, \quad F_2 = \frac{F l_1}{l_1 + l_2} \quad (4)$$

Displacements of springs k_1 and k_2 are given by

$$x_1 = \frac{F_1}{k_1} = \frac{F l_2}{k_1 (l_1 + l_2)}, \quad x_2 = \frac{F_2}{k_2} = \frac{F l_1}{k_2 (l_1 + l_2)} \quad (5)$$

Displacement of force F can be found using Eqs. (5) in Eq. (1):

$$x = \frac{l_2}{l_1+l_2} \cdot \frac{F l_2}{k_1 (l_1+l_2)} + \frac{l_1}{l_1+l_2} \cdot \frac{F l_1}{k_2 (l_1+l_2)}$$

$$= \frac{F}{(l_1 + l_2)^2} \left(\frac{l_1^2 k_1 + l_2^2 k_2}{k_1 k_2} \right) \quad (6)$$

The equivalent spring constant of the system in the

direction of x, k_e, is given by Eq. (6):

$$k_e = \frac{F}{x} = \frac{(l_1 + l_2)^2 \, k_1 k_2}{l_1^2 k_1 + l_2^2 k_2} \qquad (7)$$

1.9 Equivalence of potential energies gives

$$\frac{1}{2} k_{t1} \theta^2 + \frac{1}{2} k_{t2} \theta^2 + \frac{1}{2} k_1 (\theta l_1)^2 + \frac{1}{2} k_2 (\theta l_1)^2 + \frac{1}{2} k_3 (\theta l_2)^2 = \frac{1}{2} k_{eq} \theta^2$$

$$\therefore\ k_{eq} = k_{t1} + k_{t2} + k_1 l_1^2 + k_2 l_1^2 + k_3 l_2^2$$

1.10 k_{123} = for series springs k_1, k_2 and k_3:

$$\frac{1}{k_{123}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} \ ; \qquad k_{123} = \frac{k_1 k_2 k_3}{k_1 k_2 + k_2 k_3 + k_3 k_1}$$

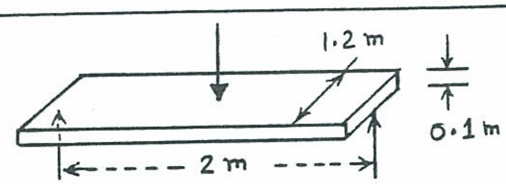
Using energy equivalence,

$$\frac{1}{2} k_{eq} \theta^2 = \frac{1}{2} k_4 \theta^2 + \frac{1}{2} k_{123} \theta^2 + \frac{1}{2} k_5 (\theta R)^2 + \frac{1}{2} k_6 (\theta R)^2$$

$$\therefore\ k_{eq} = k_4 + k_{123} + R^2 k_5 + R^2 k_6$$

$$= k_4 + \left(\frac{k_1 k_2 k_3}{k_1 k_2 + k_2 k_3 + k_3 k_1} \right) + R^2 (k_5 + k_6)$$

1.11 For simply supported beam, for load at middle,



$$k_1 = \frac{48 EI}{l^3} = \frac{48 (2.06 \times 10^{11})(10^{-4})}{8}$$

$$= 12.36 \times 10^7 \text{ N/m} \quad \text{where } I = \frac{1}{12} (1.2)(0.1)^3 = 10^{-4} \text{ m}^4.$$

$$\delta_1 = \text{original deflection} = \frac{mg}{k_1} = \frac{500 \times 9.81}{12.36 \times 10^7} = 396.8447 \times 10^{-7} \text{ m}$$

When spring k is added, $k_{eq} = k + k_1$

(a) New deflection $= \dfrac{mg}{k_{eq}} = \dfrac{\delta_1}{4}$; $k_{eq} = \dfrac{4\, mg}{\delta_1} = 4 k_1$
$= k + k_1$

$$\therefore\ k = 3 k_1 = 37.08 \times 10^7 \text{ N/m}$$

(b) New deflection $= \dfrac{mg}{k_{eq}} = \dfrac{\delta_1}{2}$; $k_{eq} = \dfrac{2\, mg}{\delta_1} = 2 k_1$
$= k + k_1$

$$\therefore\ k = k_1 = 12.36 \times 10^7 \text{ N/m}$$

(c) New deflection $= \dfrac{mg}{k_{eq}} = \dfrac{3}{4} \delta_1$; $k_{eq} = \dfrac{4\, mg}{3 \delta_1} = \dfrac{4}{3} k_1$
$= k + k_1$

$$\therefore\ k = \frac{1}{3} k_1 = 4.12 \times 10^7 \text{ N/m}$$

1.12 For a bar with length L, Young's modulus E and cross-section A, the axial stiffness (k) is given by

$$k = \frac{AE}{L} \qquad (1)$$

when cross-section is solid circular with diameter d,

$$\text{area} = A_1 = \pi d^2/4 \qquad (2)$$

when cross-section is square with side d,

$$\text{area} = A_2 = d^2 \qquad (3)$$

When cross-section is hollow circular with mean dia. d and wall thickness $t = 0.1d$,

$$\text{area} = \pi d t = \pi d(0.1 d) = 0.1 \pi d^2 \qquad (4)$$

For specified value of $k = \bar{k}$, cross-section area required is:

$$A = \frac{\bar{k} L}{E} = c \text{ (constant)} \qquad (5)$$

Weight of bar:

with solid circular section:

$$W_1 = \frac{\pi d^2}{4} L = c L \quad \text{with} \quad d^2 = \frac{4c}{\pi} \qquad (6)$$

with hollow circular section:

$$W_3 = 0.1 \pi d^2 L = 0.1 \pi \left(\frac{4c}{\pi}\right) L = 0.4 c L = 0.4 W_1 \qquad (7)$$

with square section:

$$W_2 = d^2 L = \frac{4c}{\pi} L = \frac{4}{\pi} W_1 = 1.2732 W_1 \qquad (8)$$

$\therefore$ The shaft with the hollow circular cross-section corresponds to minimum weight.

1.13 Stiffness of a cantilever beam under a bending force at free end:

$$k = \frac{3EI}{l^3} \qquad (1)$$

For a specified value of $k = \bar{k}$,

$$I = \frac{\bar{k}\, l^3}{3E} = c = \text{constant} \qquad (2)$$

For a solid circular section with diameter d,

$$I_1 = \frac{\pi d^4}{64} = c \quad \Rightarrow \quad d^4 = \frac{64\, c}{\pi} \quad \text{or} \quad d^2 = \sqrt{\frac{64\, c}{\pi}} \qquad (3)$$

weight of beam $= W_1 = \dfrac{\pi d^2 l}{4} = \dfrac{\pi l}{4}\sqrt{\dfrac{64\, c}{\pi}}$

$$= 3.5449\, l \sqrt{c} \qquad (4)$$

For a hollow circular section with mean diameter d and wall thickness $t = 0.1\, d$, weight of beam (W_2) is:

$$W_2 = \frac{\pi}{4}\left(d_o^4 - d_i^4\right) l = \frac{\pi l}{4}\left\{(d+t)^2 - (d-t)^2\right\}$$

$$= \frac{\pi l}{4}(4\, d t) = \pi\, d\, t\, l = \pi\, l\, (0.1\, d^2)$$

$$= 0.1\, \pi\, l \sqrt{\frac{64 c}{\pi}} = 1.4180\, l \sqrt{c} \qquad (5)$$

For a square section with side d, weight of the beam (W_3) is:

$$W_3 = d^2 l = l \sqrt{\frac{64\, c}{\pi}} = 4.5135\, l \sqrt{c} \qquad (6)$$

By comparing Eqs. (4), (5) and (6), the minimum weight beam corresponds to the hollow circular cross-section.

1.14 Spring force is given by $F = 800\,x + 40\,x^3$ (1)

static equilibrium of the rubber mounting (x^*) under the weight of the electronic instrument is given by

$$F = 200 = 800\,x^* + 40\,x^{*3}$$

or

$$40\,x^{*3} + 800\,x^* - 200 = 0 \qquad (2)$$

The roots of the cubic equation (2) can be found from MATLAB as

$$x^* = 0.2492, \quad -0.1246 \pm 4.4773\,i \qquad (3)$$

Thus the static equilibrium position of the rubber mounting is given by the real root of Eq. (2):

$$x^* = 0.2492 \text{ in} \qquad (4)$$

(a) Equivalent linear spring constant of rubber mounting at its static equilibrium position, using Eq. (1.7), is:

$$k_{eq} = \left.\frac{dF}{dx}\right|_{x^*} = 800 + 120\,x^{*2} = 800 + 1200\,(0.2492)^2$$

$$= 807.4521 \text{ lb/in} \qquad (5)$$

(b) Deflection of rubber mounting corresponding to the equivalent linear spring constant is:

$$k = \frac{F}{k_{eq}} = \frac{200}{807.4521} = 0.2477 \text{ in} \qquad (6)$$

1.15 $$F(x) = 200x + 50x^2 + 10x^3 \quad (1)$$

When the spring undergoes a steady deflection of $x^* = 0.5$ in during the operation of the engine, the force exerted on the spring can be found as

$$F = 200(0.5) + 50(0.5)^2 + 10(0.5)^3 = 113.75 \text{ lb} \quad (2)$$

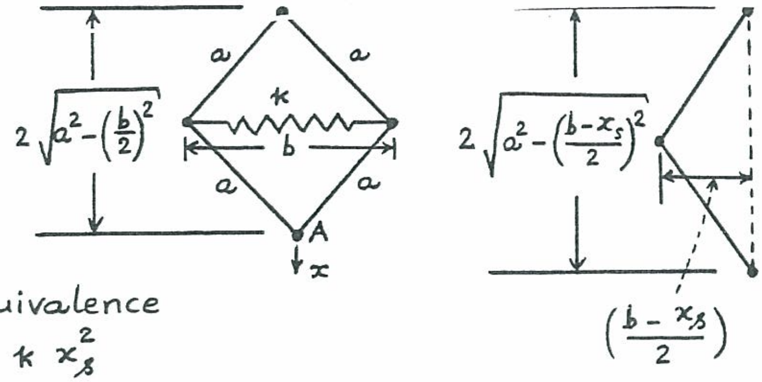
Equivalent linear spring constant at its steady deflection is given by Eq. (1.7):

$$k_{eq} = \left.\frac{dF}{dx}\right|_{x=x^*} = 200 + 100\,x^* + 30\,x^{*2}$$

$$= 200 + 100(0.5) + 30(0.5)^2$$

$$= 253.75 \text{ lb/in}$$

1.16 (a) x = downward deflection of point A,
x_s = resulting deformation of spring



Potential energy equivalence gives $\frac{1}{2} k_{eq} x^2 = \frac{1}{2} k x_s^2$

$$k_{eq} = k \left(\frac{x_s}{x}\right)^2$$

But
$$x = 2\left[\sqrt{a^2 - \left(\frac{b-x_s}{2}\right)^2} - \sqrt{a^2 - \left(\frac{b}{2}\right)^2}\right]$$

$$= 2\sqrt{a^2-\left(\frac{b}{2}\right)^2}\left[\left\{\frac{a^2 - \left\{\frac{b}{2}\left(1-\frac{x_s}{b}\right)\right\}^2}{a^2-\left(\frac{b}{2}\right)^2}\right\}^{1/2} - 1\right]$$

$$= 2\sqrt{a^2 - \frac{b^2}{4}}\left[\left\{\frac{\left(a^2 - \frac{b^2}{4} - \frac{x_s^2}{4} + \frac{b\,x_s}{2}\right)}{\left(a^2 - \frac{b^2}{4}\right)}\right\}^{1/2} - 1\right]$$

$$= 2\sqrt{a^2 - \frac{b^2}{4}}\left[\left\{1 - \frac{x_s^2 \approx 0}{\left(a^2-\frac{b^2}{4}\right)} + \frac{b\,x_s}{2\left(a^2-\frac{b^2}{4}\right)}\right\}^{1/2} - 1\right]$$

Using the relation $(1+\theta)^{1/2} \simeq 1 + \frac{\theta}{2}$, we obtain

$$x = 2\left(a^2 - \frac{b^2}{4}\right)^{1/2}\left[1 + \frac{b\,x_s}{4\left(a^2-\frac{b^2}{4}\right)} - 1\right] = \frac{b\,x_s}{2\left(a^2-\frac{b^2}{4}\right)^{1/2}}$$

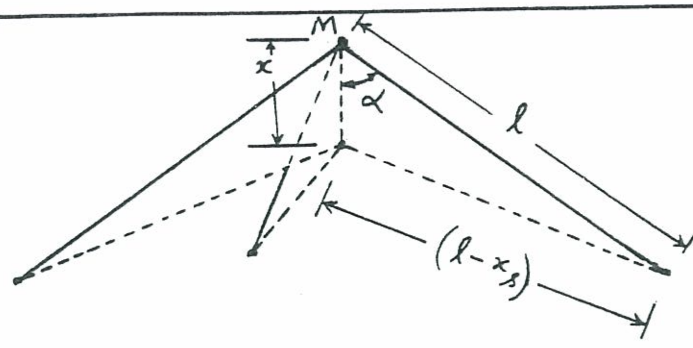
$$\therefore\ k_{eq} = k\left(\frac{x_s}{x}\right)^2 = 4k\left(\frac{a^2 - \frac{b^2}{4}}{b^2}\right) = k\left(\frac{4a^2 - b^2}{b^2}\right)$$

(b) Here $x = x_s$ (spring deflection)

$$\therefore\ k_{eq} = k$$

1.17 Let x = vertical displacement of mass M,

x_s = resulting deformation of each inclined spring.



From equivalence of potential energy,

$$\frac{1}{2} k_{eq} x^2 = 3 \left(\frac{1}{2} k x_s^2 \right) \quad ; \quad k_{eq} = 3 k \left(\frac{x_s}{x} \right)^2$$

From geometry, $$(l - x_s)^2 = l^2 + x^2 - 2 l x \cos \alpha$$

$$x^2 - 2x l \cos \alpha + 2 l x_s - x_s^2 = 0 \qquad (E_1)$$

Solving (E_1), $$x = l \cos \alpha \left[1 \pm \left\{ 1 - \frac{(2 l x_s - x_s^2)}{l^2 \cos^2 \alpha} \right\}^{1/2} \right] \qquad (E_2)$$

Using the relation $\sqrt{1-\theta} \approx 1 - \frac{\theta}{2}$, (E_2) can be rewritten as

$$x = l \cos \alpha \left[1 \pm \left\{ 1 - \left(\frac{2 l x_s - x_s^2}{2 l^2 \cos^2 \alpha} \right) \right\} \right] \qquad (E_3)$$

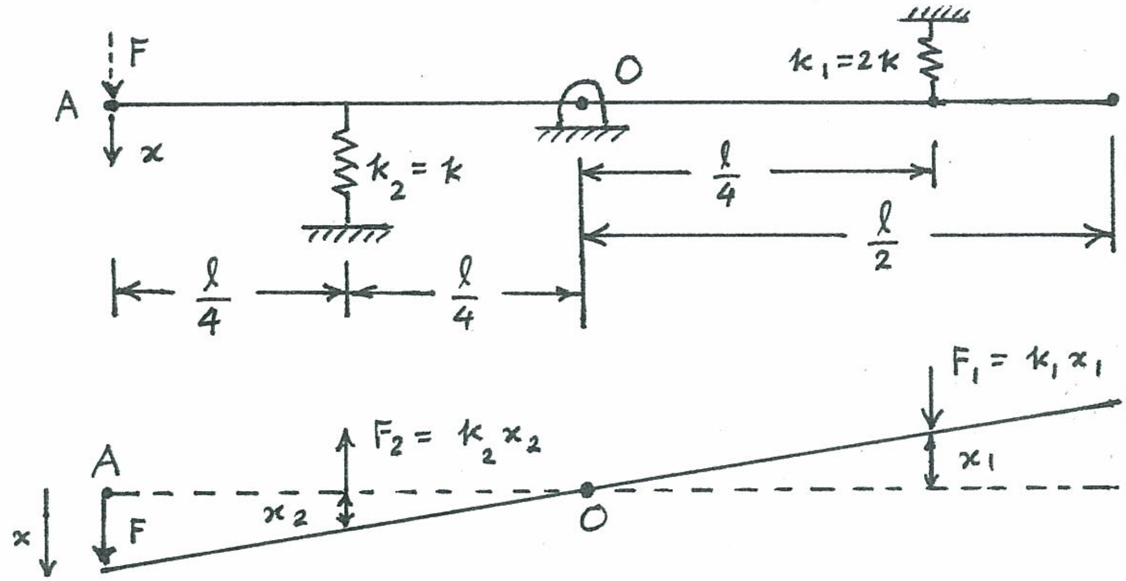
Assuming x to be small, we use minus sign and neglect x_s^2 compared to $2 l x_s$ in (E_3). This gives

$$x = \frac{x_s}{\cos \alpha}$$

$$\therefore \quad k_{eq} = 3 k \cos^2 \alpha$$

In a similar manner, $c_{eq} = 3 c \cos^2 \alpha$

1.18



$$x_2 = \frac{x}{2}, \quad x_1 = \frac{x}{2}$$

$$F_2 = k_2 x_2 = \frac{kx}{2}, \quad F_1 = k_1 x_1 = 2k\left(\frac{x}{2}\right) = kx$$

Equivalent spring constant of the system (k_{eq}) at point A can be determined by considering the moment equilibrium of forces about the pivot point O:

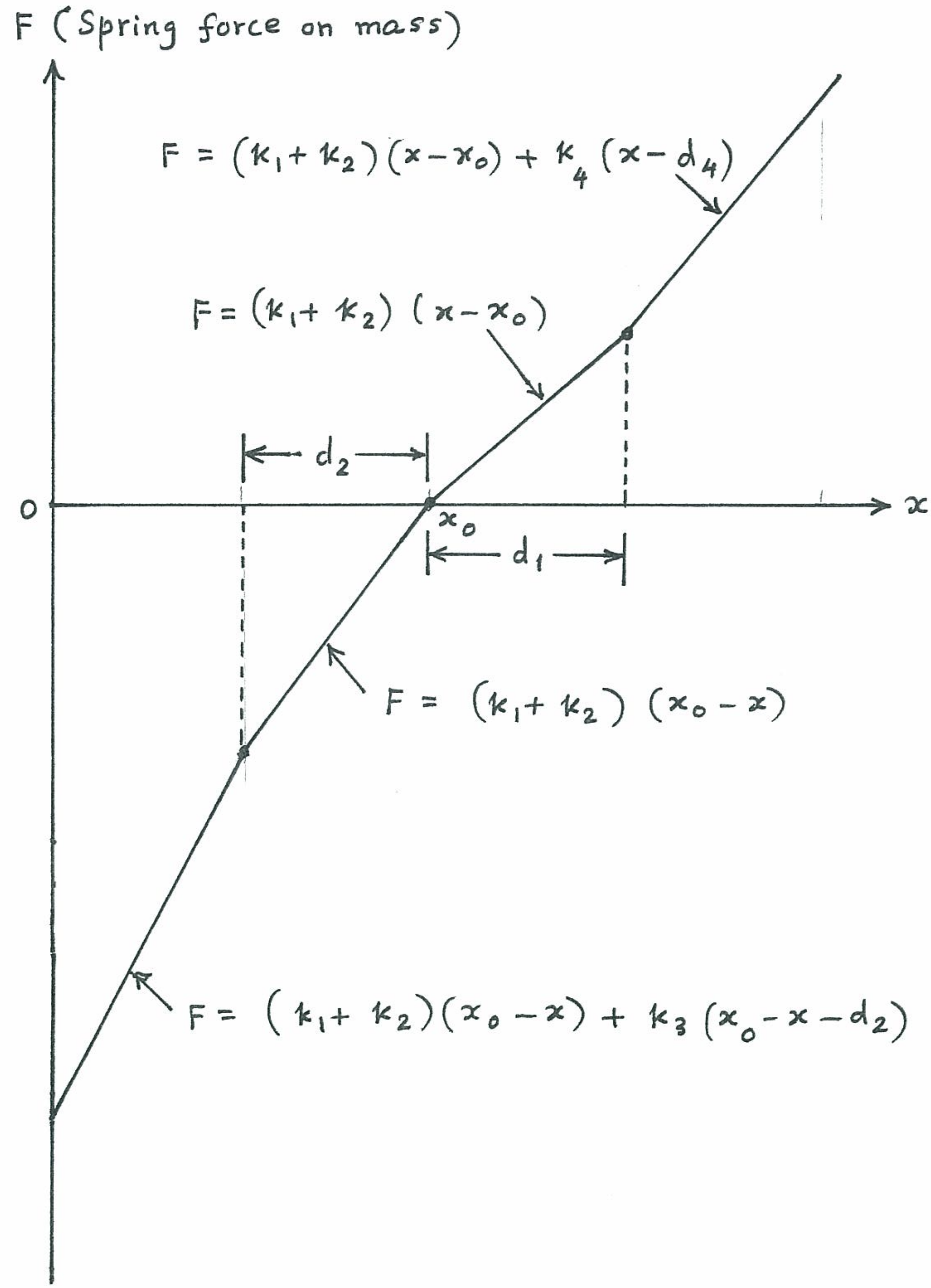
$$F\left(\frac{l}{2}\right) - F_2\left(\frac{l}{4}\right) - F_1\left(\frac{l}{4}\right) = 0$$

$$F = \frac{F_2}{2} + \frac{F_1}{2} = \frac{kx}{4} + \frac{kx}{2} = \frac{3}{4}kx$$

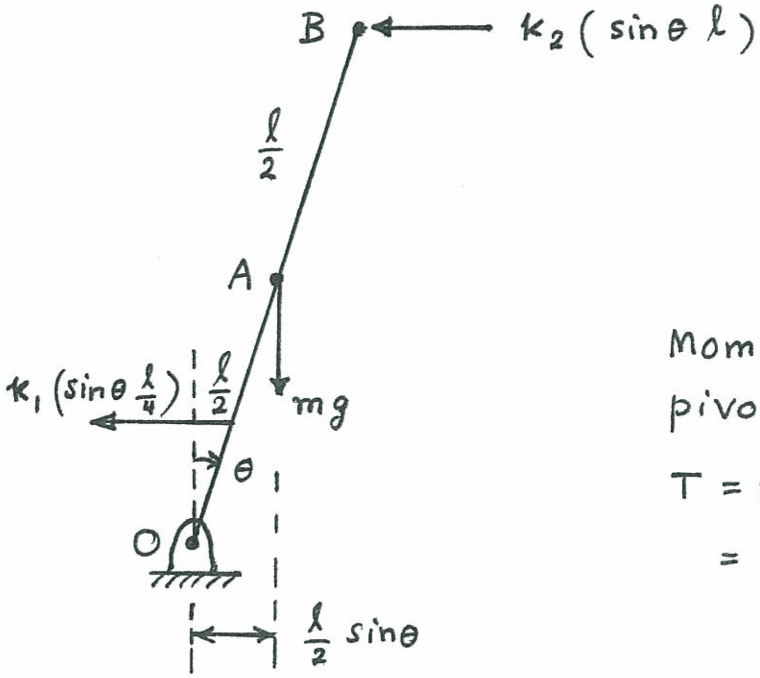
$$= k_{eq}\, x$$

$$\therefore \quad k_{eq} = \frac{3}{4}k$$

1.19


F (Spring force on mass)
$F = (k_1 + k_2)(x - x_0) + k_4 (x - d_4)$
$F = (k_1 + k_2)(x - x_0)$
d_2
0
x
x_0
d_1
$F = (k_1 + k_2)(x_0 - x)$
$F = (k_1 + k_2)(x_0 - x) + k_3 (x_0 - x - d_2)$

1.20



Moment about the pivot point O:

$$T = \text{moment}$$
$$= mg\frac{l}{2}\sin\theta - \left(k_1 \frac{l}{4}\sin\theta\right)\frac{l}{4} - (k_2\, l \sin\theta)\, l$$
$$\simeq \left(\frac{mgl}{2} - k_1\frac{l^2}{16} - k_2 l^2\right)\theta \qquad (1)$$

Denoting the equivalent torsional spring constant of the system as k_t, the moment T can be expressed as

$$T = k_t\,\theta \qquad (2)$$

By equating Eqs. (1) and (2), we obtain

$$k_t = \frac{mgl}{2} - \frac{k_1 l^2}{16} - k_2 l^2 \qquad (3)$$

1.21 When mercury is displaced by an amount x in one leg of the manometer (Fig. 1.77), the mercury column will undergo a total displacement of $2x$. The magnitude of the force, due to the weight of the displaced mercury, acts on the rest of the fluid. The restoring force is given by

$$F = 2\gamma A x \quad (1)$$

where γ is the specific weight of mercury and A is the cross-sectional area of the manometer tube. If k_{eq} denotes the spring constant associated with the restoring force, the restoring force can be expressed as

$$F = k_{eq}\, x \quad (2)$$

Equations (1) and (2) yield the equivalent spring constant as

$$k_{eq} = 2\gamma A \quad (3)$$

1.22 When the drum is displaced by an amount x from its static equilibrium position, the weight of the fluid (sea water) displaced is given by

$$W = \rho_w g \left(\frac{\pi d^2}{4}\right) x \qquad (1)$$

where ρ_w is the density of sea water and g is the acceleration due to gravity. The weight, W, given by Eq. (1) also denotes the restoring force F. By expressing the restoring force as

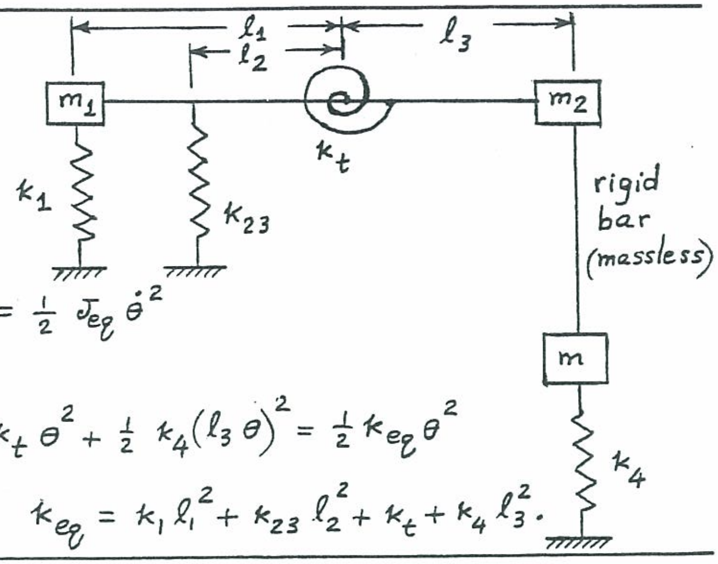
$$F = k_{eq}\, x \qquad (2)$$

where k_{eq} denotes the equivalent spring constant associated with the restoring force. Equating (1) and (2), we obtain

$$k_{eq} = \rho_w g \frac{\pi d^2}{4} \qquad (3)$$

1.23

$$k_{23} = \frac{k_2 k_3}{k_2 + k_3}$$



$$k_4 = A \rho g = \frac{\pi d^2}{4} \rho g$$

From kinetic energy,

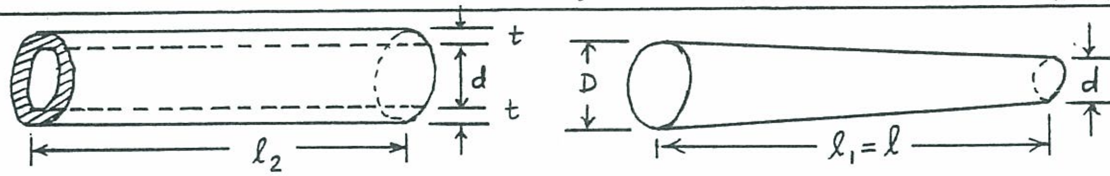
$$\frac{1}{2} m_1 (l_1 \dot{\theta})^2 + \frac{1}{2}(m_2 + m)(l_3 \dot{\theta})^2 = \frac{1}{2} J_{eq} \dot{\theta}^2$$

From potential energy,

$$\frac{1}{2} k_1 (l_1 \theta)^2 + \frac{1}{2} k_{23} (l_2 \theta)^2 + \frac{1}{2} k_t \theta^2 + \frac{1}{2} k_4 (l_3 \theta)^2 = \frac{1}{2} k_{eq} \theta^2$$

$$\therefore \quad J_{eq} = m_1 l_1^2 + (m_2 + m) l_3^2 \; ; \qquad k_{eq} = k_1 l_1^2 + k_{23} l_2^2 + k_t + k_4 l_3^2 .$$

1.24



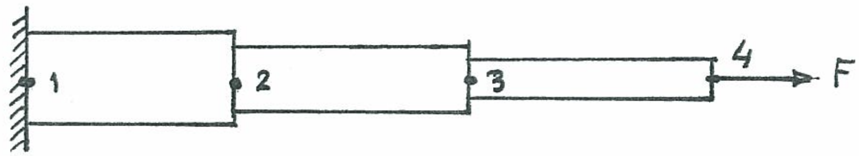
$$k_2 = \frac{EA}{l_2} = \frac{\pi E t (d+t)}{l_2} \qquad k_1 = \frac{\pi E D d}{4 l}$$

$$k_2 = k_1 \text{ gives } \quad l_2 = \frac{4 t (d+t)}{D d}$$

1.25 (a) Spring constant (stiffness) of step i in the axial direction:

$$k_i = \frac{A_i E_i}{l_i} = \frac{A_i E}{l_i} \quad , \; i = 1,2,3 \qquad (1)$$

(b)



The reaction at any point along the stepped shaft due to an axial force (F) applied at point 4 will be same as F. Hence the springs (stiffnesses) corresponding to the three steps 12, 23 and 34 are to be considered as series springs. In view of Eq.(1), the equivalent spring constant given by Eq. (1.17) becomes

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} = \frac{1}{E}\left(\frac{l_1}{A_1} + \frac{l_2}{A_2} + \frac{l_3}{A_3}\right)$$

$$= \frac{1}{E} \frac{(l_1 A_2 A_3 + l_2 A_1 A_3 + l_3 A_1 A_2)}{A_1 A_2 A_3}$$

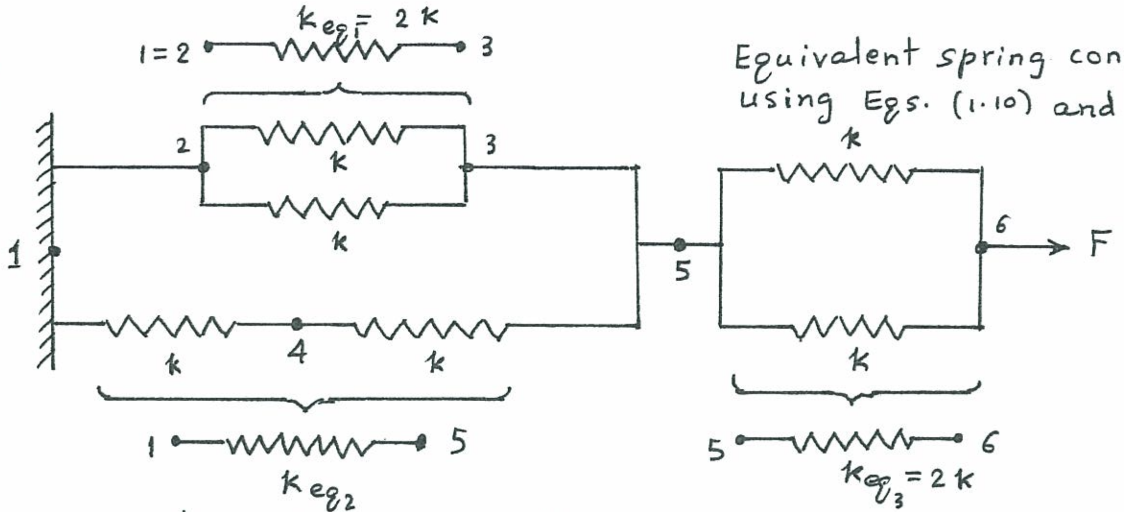
or

$$k_{eq} = \frac{E A_1 A_2 A_3}{l_1 A_2 A_3 + l_2 A_1 A_3 + l_3 A_1 A_2} \qquad (2)$$

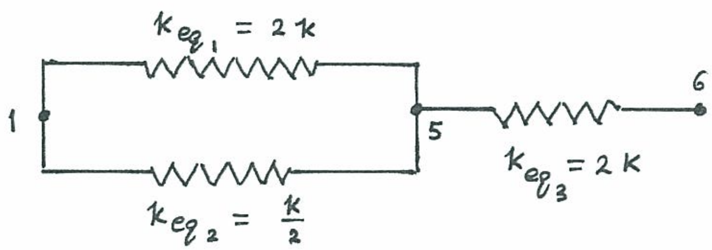
(c) steps behave as series springs.

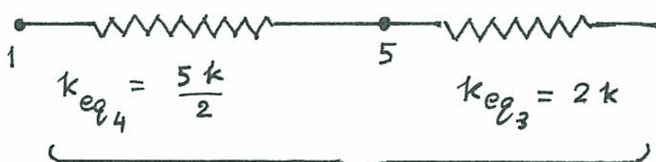
1.26

Equivalent spring constants using Eqs. (1.10) and (1.16).



$$\frac{1}{k_{eq_2}} = \frac{1}{k} + \frac{1}{k} \Rightarrow k_{eq_2} = \frac{k}{2}$$





$$k_{eq\,5} \Rightarrow \frac{1}{k_{eq_5}} = \frac{1}{k_{eq_4}} + \frac{1}{k_{eq_3}} = \frac{2}{5k} + \frac{1}{2k}$$

$$k_{eq_5} = \frac{10}{9}\,k$$

$$k_{eq} = k_{eq_5} = \frac{10}{9}\,k$$

1.27 (a) Torsional spring constant or stiffness of step i is

$$k_{t_i} = \frac{G_i J_i}{l_i} = \frac{G_i \pi D_i^4}{32\, l_i}, \quad i = 1, 2, 3 \qquad (1)$$

(b) The reactive torque at any point along the stepped shaft due to an applied torque T at the free end will be T. Hence the torsional stiffnesses (springs) corresponding to the three steps 12, 23 and 34 are to be considered as series springs. In view of Eq. (1), the equivalent torsional spring constant given by Eq. (1.17) becomes (Eq. (1.17) is to be interpreted for torsional springs):

$$\frac{1}{k_{eq}} = \frac{1}{k_{t1}} + \frac{1}{k_{t2}} + \frac{1}{k_{t3}} = \frac{32}{\pi G}\left(\frac{l_1}{D_1^4} + \frac{l_2}{D_2^4} + \frac{l_3}{D_3^4}\right)$$

$$= \frac{32}{\pi G}\left(\frac{l_1 D_2^4 D_3^4 + l_2 D_1^4 D_3^4 + l_3 D_1^4 D_2^4}{D_1^4 D_2^4 D_3^4}\right)$$

or

$$k_{eq} = \frac{\pi G D_1^4 D_2^4 D_3^4}{32\left(l_1 D_2^4 D_3^4 + l_2 D_1^4 D_3^4 + l_3 D_1^4 D_2^4\right)} \qquad (2)$$

(c) steps behave as series springs.

1.28 (a) $F \simeq F\big|_{x_0} + \frac{dF}{dx}\Big|_{x_0} \cdot (x - x_0) = \left(500\,x + 2x^3\right)_{x=10} + \left(500 + 6x^2\right)_{x=10} \cdot (x-10)$

$\simeq 1100\,x - 4000$

(b) at $x = 9$ mm :

Exact $F_9 = 500 \times 9 + 2\,(9)^3 = 5958$ N

Approximate $F_9 = 1100 \times 9 - 4000 = 5900$ N

Error $= -0.9735\ \%$

(c) at $x = 11$ mm :

Exact $F_{11} = 500 \times 11 + 2\,(11)^3 = 8162$ N

Approximate $F_{11} = 1100 \times 11 - 4000 = 8100$ N

Error $= +0.7596\ \%$

1.29 $p v^\gamma$ = constant --- (E_1) ; Differentiation of (E_1) gives

$$dp\ v^\gamma + p\,\gamma\, v^{\gamma-1}\, dv = 0$$

$$dp = -\frac{p\gamma}{v}\, dv \quad ---(E_2)$$

change in volume when mass moves by dx, $dv = -A \cdot dx$ --- (E_3)

Eqs. (E_2) and (E_3) give $dp = \frac{p\gamma A}{v}\, dx$

Force due to pressure change $= dF = dp \cdot A = \frac{p\gamma A^2}{v} \cdot dx$

spring constant of air spring $= k = \frac{dF}{dx} = \left(\frac{p\gamma A^2}{v}\right)$.

1.30 Equivalent spring constants in diffent directions are

$k_{e1} = \left(\frac{k_5\ k_6\ k_7}{k_5 k_6 + k_5 k_7 + k_6 k_7}\right)$, $\quad k_{e2} = \left(\frac{k_8\ k_9}{k_8 + k_9}\right)$,

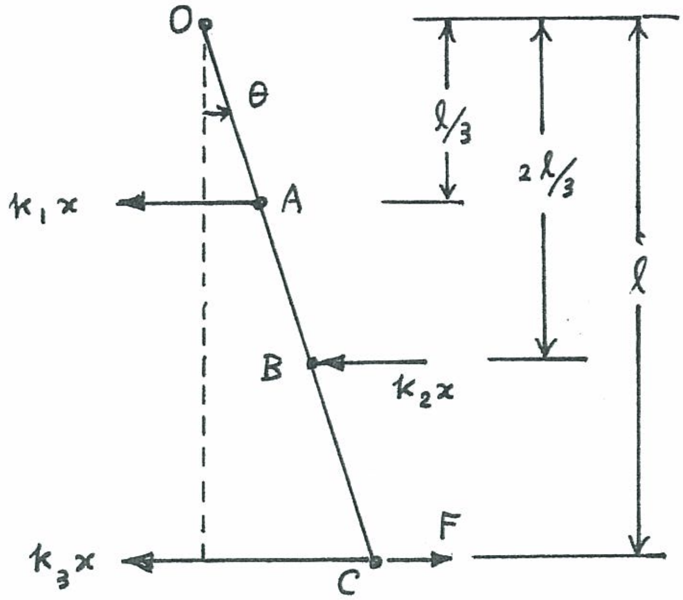
$k_{e3} = \left(\frac{k_1\ k_2}{k_1 + k_2}\right)$, $\quad k_{e4} = \left(\frac{k_3\ k_4}{k_3 + k_4}\right)$

If the force P moves by x, spring located at θ_i undergoes a displacement of $x_i = x \cos\theta_i$ (derivation as in problem 1.17).

Equivalence of potential energy gives $\frac{1}{2} k_{eq}\, x^2 = \frac{1}{2} \sum_{i=1}^{4} k_{ei}\, x_i^2$

$$k_{eq} = \sum_{i=1}^{4} \left(k_{ei}\ \cos^2\theta_i\right)$$

1.31



Let the link OABC undergo a small angular displacement θ as shown in above figure. The spring reaction forces are also indicated in the figure. Equilibrium of moments about the pivot point O gives:

$$-k_1 x\left(\frac{l}{3}\right) - k_3 x\,(l) - k_2 x\left(\frac{2l}{3}\right) + F\,(l) = 0$$

or
$$F = \left(\frac{k_1}{3} + \frac{2}{3}k_2 + k_3\right)x \qquad (1)$$

If k_{eq} denotes the equivalent spring constant of the link along the direction of F at point C, we have

$$F = k_{eq}\, x \qquad (2)$$

Equations (1) and (2) give

$$k_{eq} = \frac{k_1}{3} + \frac{2k_2}{3} + k_3 = \frac{k}{3} + \frac{2}{3}(2k) + (3k)$$

$$\therefore\ k_{eq} = \frac{14}{3}k \qquad (3)$$

1.32 Spring constant of a helical spring is

$$k = \frac{G d^4}{8 N D^3} \qquad (1)$$

Assuming the shear modulus of steel as $G = 79.3$ GPa, Eq. (1) gives, for $D = 0.2$ m, $d = 0.005$ m and $N = 10$,

$$k = \frac{(79.3 \times 10^9)(0.005)^4}{8\,(10)\,(0.2)^3} = 77.4414 \text{ N/m}$$

1.33 (a) D and d: same for both helical springs

Weight of a helical spring is:

$$W = \pi D \left(\frac{\pi d^2}{4}\right) N \gamma \qquad (1)$$

where γ = specific weight of material of spring.

For a steel spring with $\gamma_s = 76.5\ kN/m^3$, the weight is (for $N_s = 10$):

$$W_s = \pi D \left(\frac{\pi d^2}{4}\right) N_s \gamma_s = \frac{\pi^2 D d^2}{4} (10)(76.5 \times 10^3)$$

$$= 19.125 \times 10^4\ \pi^2 D d^2 \qquad (2)$$

For an aluminum spring with $\gamma_a = 26.6\ kN/m^3$, the weight is (for number of turns N_a),

$$W_a = \pi D \left(\frac{\pi d^2}{4}\right) N_a \gamma_a = \frac{\pi^2 D d^2 N_a}{4} (26.6 \times 10^3)$$

$$= 6.65 \times 10^3\ \pi^2 D d^2 N_a \qquad (3)$$

Equating (2) and (3),

$$19.125 \times 10^4\ \pi^2 D d^2 = 6.65 \times 10^3\ \pi^2 D d^2 N_a$$

or

$$N_a = \frac{19.125 \times 10^4}{6.65 \times 10^3} = 28.7594 \qquad (4)$$

(b) Spring constant of a helical spring is:

$$k = G d^4 / (8 N D^3)$$

For a steel spring with $G = 793.3$ GPa,

$$k_s = (79.3 \times 10^9)\, d^4 / \{8 (10) D^3\}$$

$$= 0.99125 \times 10^9\ d^4 / D^3 \qquad (5)$$

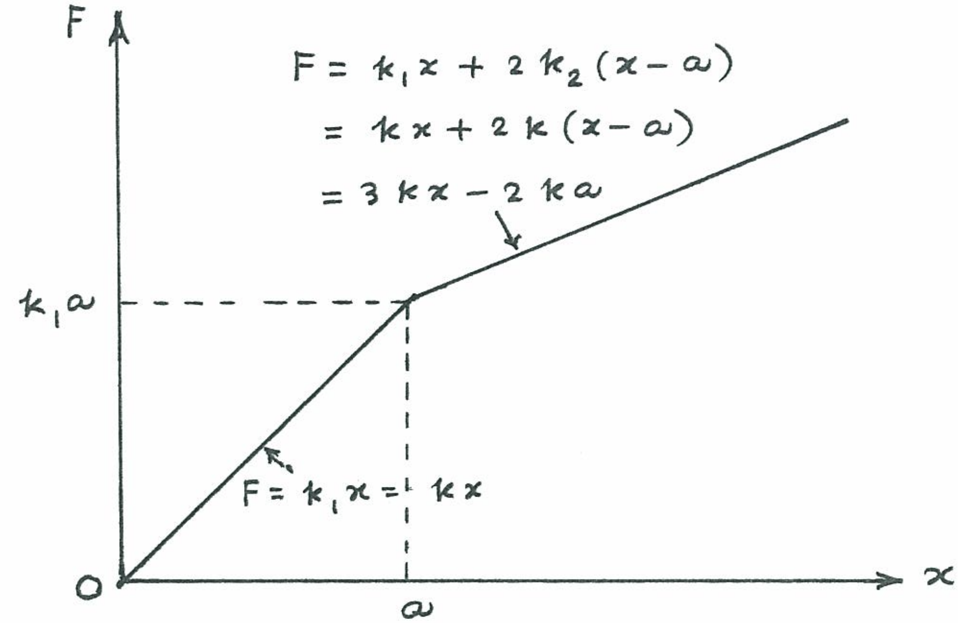
For an aluminum spring with $G = 26.2$ GPa,

$$k_a = (26.2 \times 10^9)\, d^4 / \{8 (28.7594) D^3\}$$

$$= 0.1139 \times 10^9\ d^4 / D^3 \qquad (6)$$

Eqs. (5) and (6) indicate that the spring constant of steel spring is $0.99125/0.1139 = 8.7046$ times larger than that of aluminum spring.

1.34



1.35 From Problem 1.29, $k = \dfrac{p\gamma A^2}{v}$ with $\gamma = 1.4$ for air

Let $p = 200$ psi

$k = 75$ lb/in $= \dfrac{(200)(1.4)A^2}{v} \Rightarrow \dfrac{A^2}{v} = 0.2679$

Let diameter of piston $= d = 2$ inch ; $A = \dfrac{\pi}{4}(2)^2 = 3.1416$ in^2

$v = A^2/0.2679 = 36.8408$ in^3

Let $h = 2$ inch ; $\dfrac{\pi}{4} D^2 (2) = v \Rightarrow D = 4.8429$ inch

---

1.36 $F = a\,x + b\,x^3 = 2\,(10^4)\,x + 4\,(10^7)\,x^3$

Around x^*: $F(x) \approx F(x^*) + \dfrac{dF}{dx}\Big|_{x^*}(x - x^*)$

When $x^* = 10^{-2}$ m, $F(x^*) = 2\,(10^4)\,(10^{-2}) + 4\,(10^7)\,(10^{-6}) = 240$ N

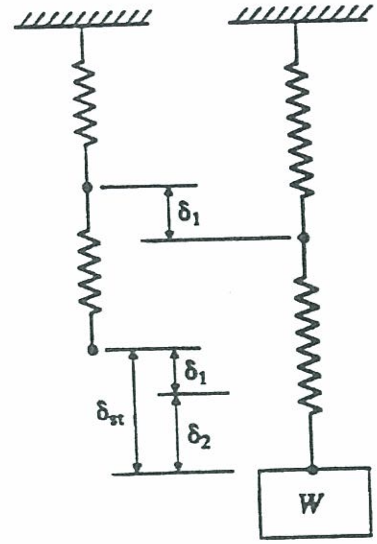
$\dfrac{dF}{dx}\Big|_{x^*} = a + 3\,b\,x^2 = 2\,(10^4) + 3\,(4)\,(10^7)\,(10^{-4}) = 32000$

Hence $F(x) = 240 + 32000\,(x - 0.01) = (32000\,x - 80)$ N

Since the linearized spring constant is given by $F(x) = k_{eq}\,x$, we have $k_{eq} = 32{,}000$ N/m.

1.37 $F_i = a_i\, x_i + b_i\, x_i^3 \quad ; \; i = 1, 2$

Springs in series:



$$W = a_1\,\delta_1 + b_1\,\delta_1^3 \qquad (1)$$

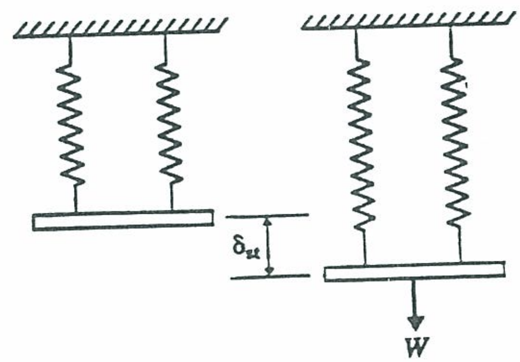
$$W = a_2\,\delta_2 + b_2\,\delta_2^3 \qquad (2)$$

$$W = k_{eq}\,\delta_{st} \qquad (3)$$

$$\delta_{st} = \delta_1 + \delta_2 \qquad (4)$$

Solve Eqs. (1) and (2) for δ_1 and δ_2, respectively. Substitute the result in Eq. (4) and then in Eq. (3) to find k_{eq}.

Springs in parallel:



$$W = F_1 + F_2$$

$$= a_1\,\delta_{st} + b_1\,\delta_{st}^3 + a_2\,\delta_{st} + b_2\,\delta_{st}^3$$

$$= k_{eq}\,\delta_{st}$$

$$k_{eq} = a_1 + b_1\,\delta_{st}^2 + a_2 + b_2\,\delta_{st}^2$$

1.38 $k = \dfrac{G\,d^4}{8\,D^3\,N} \geq 8\times10^6$ N/m ; $\dfrac{D}{d} \geq 6$; $N \geq 10$

$W = \pi D N \rho \left(\dfrac{\pi d^2}{4}\right)$ where ρ = weight per unit volume

$$f_1 = \frac{1}{2}\sqrt{\frac{kg}{W}} = \frac{1}{2}\sqrt{\frac{G\,d^2\,g}{2\pi^2 D^4 N^2 \rho}} \geq 0.4 \text{ Hz}$$

Using $G = 73.1\times10^9$ N/m^2, $\rho = 76000$ N/m^3, $g = 9.81$ m/sec^2, $\dfrac{D}{d} = 6, 8, 10$; $N = 10, 15, 20$; $d = 0.4, 0.6, \ldots,$ values of k and f_1 are computed.

Combination of $\dfrac{D}{d} = 6$, $N = 10$ and $d = 2.0$ m, corresponding to $k = 8.4606\times10^6$ N/m and $f_1 = 0.4801$ Hz, can be taken as an acceptable design.

1.39 Total elongation (strain) is same in each material:

$$\epsilon_s = \epsilon_a = \frac{x}{\ell} \tag{1}$$

where x is the total elongation. Equation (1) can be expressed as

$$\frac{\sigma_s}{E_s} = \frac{\sigma_a}{E_a} = \frac{x}{\ell} \tag{2}$$

$$\text{or} \quad \sigma_s = \frac{E_s\,x}{\ell} \tag{3}$$

$$\sigma_a = \frac{E_a\,x}{\ell} \tag{4}$$

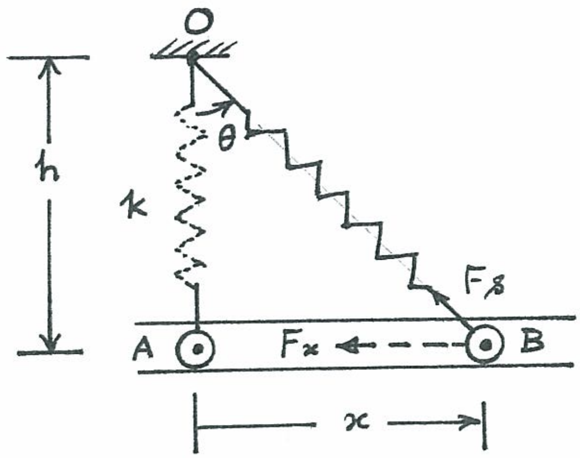
Total axial force is:

$$F = F_s + F_a = \sigma_s\,A_s + \sigma_a\,A_a \tag{5}$$

where F_s and F_a denote the axial forces acting on steel and aluminum, respectively, and A_s and A_a represent the cross-sectional areas of the two materials. Equating F to k_{eq} x where k_{eq} denotes the equivalent spring constant of the bimetallic bar, we obtain from Eqs. (3) to (5):

$$F = k_{eq}\,x = \left(\frac{E_s\,x}{\ell}\right) A_s + \left(\frac{E_a\,x}{\ell}\right) A_a$$

$$\text{or} \quad k_{eq} = \frac{E_s\,A_s}{\ell} + \frac{E_a\,A_a}{\ell} \tag{6}$$



1.40 Let the length of the spring be h. Spring is undeformed at $\theta = 0$. When the end A of the spring is displaced by an amount x as shown in the figure, the spring is stretched by the amount $\left(\sqrt{h^2+x^2} - h\right)$ so that the force in the spring (F_s) is given by

$$F_s = k\left(\sqrt{h^2+x^2} - h\right) \qquad (1)$$

The component of the spring force F_s along the direction of x is given by

$$F_x = F_s \sin\theta = F_s \frac{x}{\sqrt{h^2+x^2}} = \frac{k\left(\sqrt{h^2+x^2} - h\right)x}{\sqrt{h^2+x^2}}$$

$$= k\left(1 - \frac{h}{\sqrt{h^2+x^2}}\right)x \qquad (2)$$

Equation (2) shows that the force - displacement relation (in the x- direction) is nonlinear. If the relation is linear, we could write

$$F_x = \underset{\sim}{k}\, x \qquad (3)$$

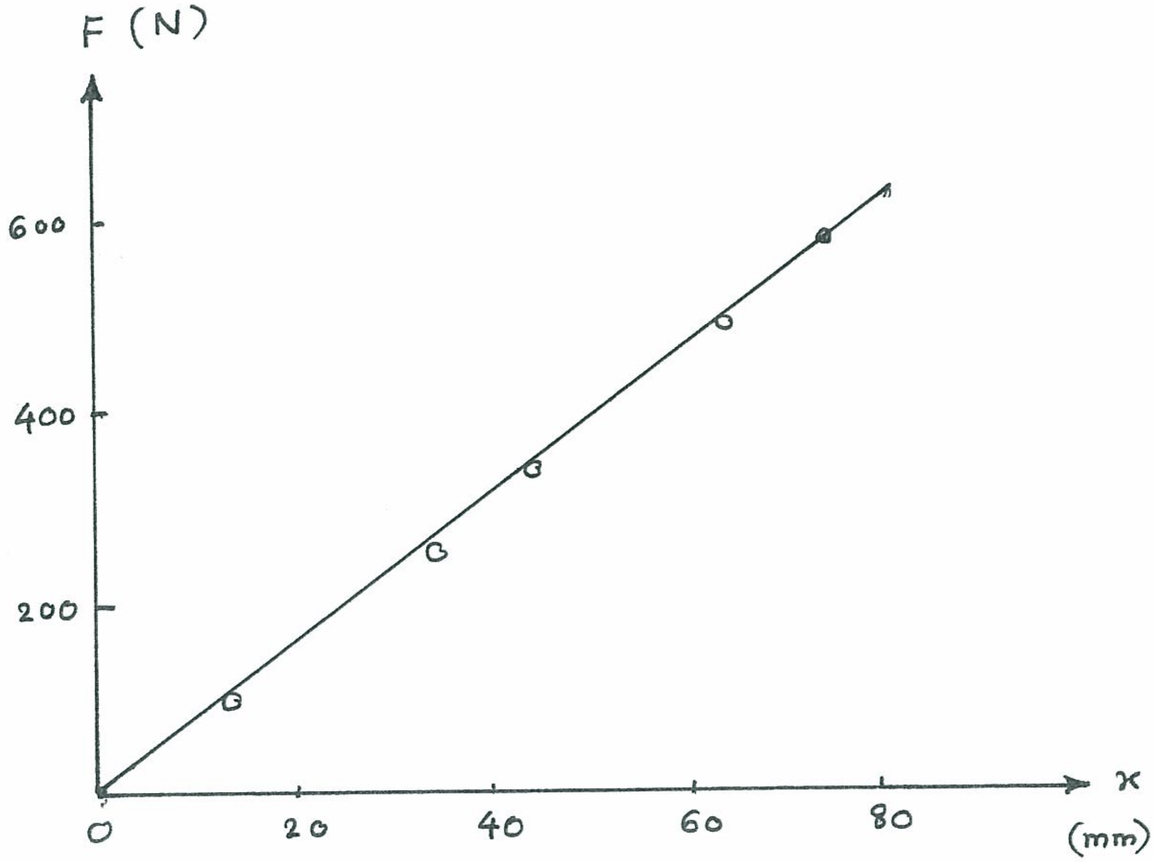
A comparison of Eqs. (2) and (3) shows that the spring constant $\underset{\sim}{x}$ is not a constant, but depends on the displacement x.

1.41 From the given data, the force - deformation relation of the spring can be obtained as :

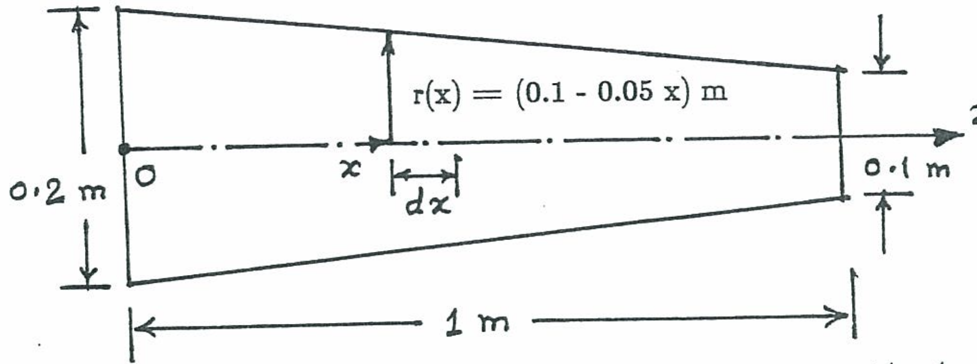
Tensile force (F), N	0	100	250	330	480	570
Deformation of spring (x), mm (change in length)	0	13	33	44	64	76

The force - deformation relation is plotten in the figure shown below. The relation can be seen to be nearly linear with the spring constant given by

$$k = \frac{F}{x} \simeq \frac{570}{76} = 7.5 \text{ N/mm} = 7500 \text{ N/m}.$$



1.42



$J = \frac{\pi}{2} r^4$ = area polar moment of inertia at section $x = 1.5708\,(0.1 - 0.05\,x)^4\ m^4$

Knowing that the angle of twist, θ, between the ends of a uniform shaft of length ℓ under a torque T is given by $\theta = \frac{T\,\ell}{GJ}$, the angle of twist for an element of length dx can be expressed as

$$d\theta = \frac{T\,dx}{GJ} = \frac{T\,dx}{(80\,(10^9))\,1.5708\,(0.1 - 0.05\,x)^4} \qquad (1)$$

The total angle of twist can be determined by integrating Eq. (1) from x=0 to 1 as:

$$\theta = \int_0^1 \frac{T\,dx}{(12.5664\,(10^{10}))\,(0.1 - 0.05\,x)^4} = \left(\frac{T}{12.5664\,(10^{10})}\right) \int_0^1 \frac{dx}{(0.1 - 0.05\,x)^4} \qquad (2)$$

$$\text{But} \int_0^1 \frac{dx}{(0.1 - 0.05\,x)^4} = -\frac{1}{0.05} \int_0^1 \frac{(-\,0.05\,dx)}{(0.1 - 0.05\,x)^4} = -\,20 \int_0^{-0.05} \frac{dy}{(0.1 + y)^4}$$

$= 4.6667\,(10^4)$ where $y = -\,0.05\,x$

$$\text{Hence}\ \ \theta = \frac{T\,(4.6667)\,(10^4)}{12.5664\,(10^{10})} = T\,(0.3714\,(10^{-6}))\ \text{rad}$$

This gives $k_t = \frac{T}{\theta} = 2.6925\,(10^6)$ N-m/rad

1.43 The steel and aluminum hollow shafts can be treated as two torsional springs in parallel. For a hollow shaft,

$$k_t = \frac{\pi\,G}{32\,\ell}\,(D^4 - d^4)$$

For the steel shaft, $G = 80\,(10^9)$ Pa, $\ell = 5$ m, $D = 0.25$ m, $d = 0.15$ m, and hence

$$k_{t_1} = \frac{\pi\,(8\,(10^{10}))}{32\,(5)}\,(0.25^4 - 0.15^4) = 5.34072\,(10^6)\ \text{N–m/rad}$$

(a) For the aluminum shaft, $G = 26\,(10^9)$ Pa, $\ell = 5$ m, $D = 0.15$ m, $d = 0.1$ m, and hence

$$k_{t_2} = \frac{\pi\,(26\,(10^9))}{32\,(5)}\,(0.15^4 - 0.10^4) = 0.207395\,(10^6)\ \text{N–m/rad}$$

$$k_{eq} = k_{t_1} + k_{t_2} = 5.34072\,(10^6) + 0.20739\,(10^6) = 5.54811\,(10^6)\ \text{N–m/rad}$$

(b) With $G = 26\,(10^9)$ Pa, $\ell = 5$ m, $D = 0.15$ m and $d = 0.05$ m,

$$k_{t_2} = \frac{\pi\ (26 \times 10^9)}{32\ (5)} \left(0.15^4 - 0.05^4\right) = 0.255255 \times 10^6 \text{ N-m/rad}$$

$$k_{eq} = k_{t_1} + k_{t_2} = 5.34072 \times 10^6 + 0.255255 \times 10^6 = 5.595975 \times 10^6 \text{ N-m/rad}$$

1.44 For helical spring: $k = \dfrac{G\ d^4}{64\ n\ R^3}$

Spring 1: $k_1 = \dfrac{(12 \times 10^6)(2^4)}{64\ (10)(6^3)} = 1{,}388.89$ lb/in

Spring 2: $k_2 = \dfrac{(4 \times 10^6)(1^4)}{64\ (10)(5^3)} = 50.00$ lb/in

(a) Spring 2 inside spring 1 (parallel): $k_{eq} = k_1 + k_2 = 1{,}438.89$ lb/in

(b) Spring 2 on top of spring 1 (series):

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} = \frac{k_2 + k_1}{k_1\ k_2}$$

which gives $k_{eq} = 48.2625$ lb/in.

1.45 For a helical spring, $k = \dfrac{G\ d^4}{64\ n\ R^3}$

$$k_1 = \frac{(12 \times 10^6)(1)^4}{64\ (10)\ (6^3)} = 86.806 \text{ lb/in}$$

$$k_2 = \frac{(4 \times 10^6)(0.5)^4}{64\ (10)\ (5^3)} = 3.125 \text{ lb/in}$$

(a) Spring 2 inside spring 1: $k_{eq} = k_1 + k_2 = 89.931$ lb/in

(b) Spring 2 on top of spring 1: $\dfrac{1}{k_{eq}} = \dfrac{1}{k_1} + \dfrac{1}{k_2}$

or $k_{eq} = \dfrac{k_1\ k_2}{k_1 + k_2} = \dfrac{86.806\ (3.125)}{86.806 + 3.125} = 3.0164$ lb/in

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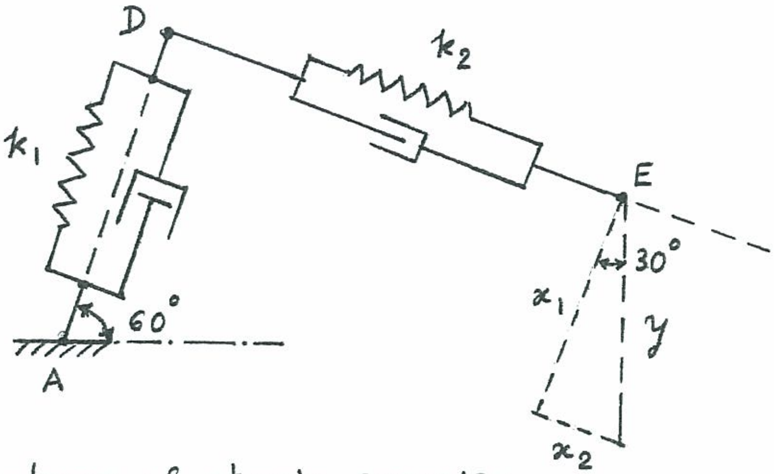
1.46



$x_2 = y \sin 30°$

$x_1 = y \cos 30°$

Equivalence of strain energies:

$$\frac{1}{2} k_{eq} y^2 = \frac{1}{2} k_2 x_2^2 + \frac{1}{2} k_1 x_1^2 = \frac{1}{2} k_1 y^2 \cos^2 30° + \frac{1}{2} k_2 y^2 \sin^2 30°$$

i.e., $k_{eq} = \frac{3}{4} k_1 + \frac{1}{4} k_2$

with $k_1 = \frac{A_1 E_1}{l_1} = \frac{\pi}{4} \frac{(10^2 - 9.5^2)(30 \times 10^6)}{100} = 2.297295 \times 10^6$ lb/in

and $k_2 = \frac{A_2 E_2}{l_2} = \frac{\pi}{4} \frac{(7^2 - 6.5^2)(30 \times 10^6)}{75} = 2.12058 \times 10^6$ lb/in

$$\therefore k_{eq} = \frac{3}{4}(2.297295 \times 10^6) + \frac{1}{4}(2.12058 \times 10^6)$$

$$= 2.25311625 \times 10^6 \text{ lb/in}$$

Similarly, the equivalent damping constant can be found as (using equivalence of kinetic energies):

$$c_{eq} = \frac{3}{4} c_1 + \frac{1}{4} c_2 = \frac{3}{4}(0.4) + \frac{1}{4}(0.3) = 0.375 \text{ lb-sec/in.}$$

1.47 stainless steel: $E = 30 \times 10^6$ lb/in², $G = 11.5 \times 10^6$ lb/in²

For each tube:

$D = 0.30''$, $d = 0.29''$, $l = 50''$

Axial stiffness $= \frac{AE}{l} = \frac{\pi}{4}(D^2 - d^2)\frac{E}{l}$

$$= \frac{\pi}{4}(0.30^2 - 0.29^2)\left(\frac{30 \times 10^6}{50}\right) = 2780.316 \text{ lb/in} = k_a$$

$$\text{Torsional stiffness} = \frac{\pi G}{32\, l} \left(D^4 - d^4 \right)$$

$$= \frac{\pi \left(11.5 \times 10^6 \right)}{32\,(50)} \left(0.30^4 - 0.29^4 \right) = 23.1942 \text{ lb-in/rad} = k_t$$

For heat exchanger with 6 tubes:

Axial stiffness $= 6\, k_a = 16{,}681.896$ lb/in

Torsional stiffness $= 6\, k_t = 139.1652$ lb-in/rad

1.48 Assume small angles θ_1 and θ_2; $\quad \theta_2 = \left(\frac{p_1}{p_2} \right) \theta_1$

x_1 = horizontal displacement of C.G. of mass $m_1 = \theta_1 r_1$

x_2 = vertical displacement of C.G. of mass $m_2 = \theta_2 r_2 = p_1 \theta_1 r_2 / p_2$

y_1 = horizontal displacement of springs k_1 and $k_2 = \theta_1 (r_1 + l_1)$

y_2 = vertical displacement of springs k_3 and $k_4 = \theta_2 l_2 = p_1 l_2 \theta_1 / p_2$

Equivalence of kinetic energies gives

$$\frac{1}{2} J_{eq} \left(\dot{\theta}_1 \right)^2 = \frac{1}{2} J_1 \left(\dot{\theta}_1 \right)^2 + \frac{1}{2} J_2 \left(\dot{\theta}_2 \right)^2 + \frac{1}{2} m_1 \left(\dot{x}_1 \right)^2 + \frac{1}{2} m_2 \left(\dot{x}_2 \right)^2$$

$$\therefore J_{eq} = J_1 + J_2 \left(p_1 / p_2 \right)^2 + m_1 r_1^2 + m_2 r_2^2 \left(p_1 / p_2 \right)^2$$

Equivalence of potential energies gives

$$\frac{1}{2} k_{eq} \theta_1^2 = \frac{1}{2} k_{12} y_1^2 + \frac{1}{2} k_{34} y_2^2 + \frac{1}{2} k_{t1} \theta_1^2 + \frac{1}{2} k_{t2} \theta_2^2$$

with $k_{12} = k_1 + k_2$, $\quad k_{34} = k_3 k_4 / (k_3 + k_4)$

$y_1 = \theta_1 (r_1 + l_1)$, $y_2 = p_1 l_2 \theta_1 / p_2$ and $\theta_2 = p_1 \theta_1 / p_2$.

$$\therefore \quad k_{eq} = (k_1 + k_2)(r_1 + l_1)^2 + \left(\frac{k_3 k_4}{k_3 + k_4} \right) \frac{p_1^2 l_2^2}{p_2^2} + k_{t1} + k_{t2} \frac{p_1^2}{p_2^2}.$$

1.49 $\theta = \frac{x}{b}$, $\quad x_1 = \frac{x a}{b}$.

From equivalence of kinetic energies,

$$\frac{1}{2} m_{eq} \dot{x}^2 = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}^2 + \frac{1}{2} J_0 \dot{\theta}^2$$

$$m_{eq} = m_1 \left(\frac{a}{b} \right)^2 + m_2 + J_0 \left(\frac{1}{b} \right)^2$$

1.50 Let $\dot{\theta}_i$ = angular velocity of the motor (input)

Angular velocities of different gear sets are:

J_{motor}, J_1	J_2, J_3	J_4, J_5	$\cdots$	J_{2N}, J_{load}
$\dot{\theta}_i$	$\dot{\theta}_i\left(\frac{n_1}{n_2}\right)$	$\dot{\theta}_i\left(\frac{n_1}{n_2}\frac{n_3}{n_4}\right)$		$\dot{\theta}_i\left(\frac{n_1}{n_2}\frac{n_3}{n_4}\cdots\frac{n_{2N-1}}{n_{2N}}\right)$

Equivalence of kinetic energies gives

$$\frac{1}{2} J_{eq}\,\dot{\theta}_i^2 = \frac{1}{2} J_{motor}\,\dot{\theta}_i^2 + \frac{1}{2}\sum_{k=1}^{2N} J_k\,\dot{\theta}_k^2 + \frac{1}{2} J_{load}\,\dot{\theta}_{load}^2$$

$$\therefore J_{eq} = (J_{motor} + J_1) + (J_2 + J_3)\left(\frac{n_1}{n_2}\right)^2 + (J_4 + J_5)\left(\frac{n_1}{n_2}\frac{n_3}{n_4}\right)^2 + \cdots + (J_{2N} + J_{load})\left(\frac{n_1}{n_2}\frac{n_3}{n_4}\cdots\frac{n_{2N-1}}{n_{2N}}\right)^2$$

1.51 Equivalence of kinetic energies gives

$$\frac{1}{2} J_{eq}\,\dot{\theta}_1^2 = \frac{1}{2} J_1\,\dot{\theta}_1^2 + \frac{1}{2} J_2\,\dot{\theta}_2^2 \quad \text{where} \quad \dot{\theta}_2 = \dot{\theta}_1\left(\frac{n_1}{n_2}\right)$$

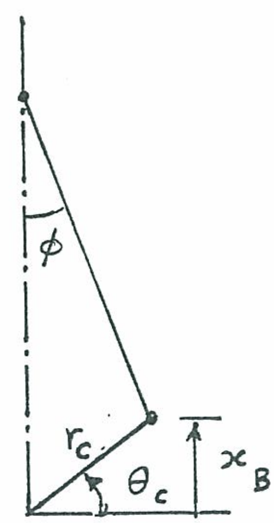
$$J_{eq} = J_1 + J_2\left(\frac{n_1}{n_2}\right)^2$$

1.52 When point A moves by distance $x = x_h$, the walking beam rotates by the angle $\theta_b = \frac{x_h}{\ell_3}$.

This corresponds to a linear motion of point B: $x_B = \theta_b\,\ell_2 = \frac{x_h\,\ell_2}{\ell_3}$

and the angular rotation of crank can be found from the relation:

$$x_B = r_c \sin\theta_c + \ell_4\cos\phi = r_c\sin\theta_c + \ell_4\sqrt{1 - \frac{r_c^2}{\ell_4^2}\sin^2\theta_c}$$



For large values of ℓ_4 compared to r_c and for small values of x and θ_c, we have

$$x_B \approx r_c\sin\theta_c = r_c\,\theta_c \text{ or } \theta_c = \frac{x_B}{r_c} = \frac{x_h\,\ell_2}{\ell_3\,r_c}$$

The kinetic energy of the system can be expressed as

$$T = \frac{1}{2} m_h\,\dot{x}_h^2 + \frac{1}{2} J_b\,\dot{\theta}_b^2 + \frac{1}{2} J_c\,\dot{\theta}_c^2$$

Equating this to $T = \frac{1}{2} m_{eq}\,\dot{x}^2 = \frac{1}{2} m_{eq}\,\dot{x}_h^2$, we obtain

$$m_{eq} = m_h + \frac{J_b}{\ell_3^2} + J_c\left(\frac{\ell_2}{\ell_3\,r_c}\right)^2.$$

1.53 When mass m is displaced by x, the bell crank lever rotates by the angle $\theta_b = \frac{x}{\ell_1}$. This makes the center of the sphere displace by $x_s = \theta_b\, \ell_2$. Since the sphere rotates with out slip, it rotates by an angle

$$\theta_s = \frac{x_s}{r_s} = \frac{\theta_b\, \ell_2}{r_s} = \frac{x\, \ell_2}{\ell_1\, r_s}$$

The kinetic energy of the system can be expressed as

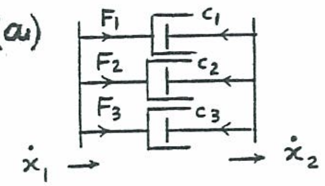
$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_0 \dot{\theta}^2 + \frac{1}{2} J_s \dot{\theta}_s^2 + \frac{1}{2} m_s \dot{x}_s^2$$

$$= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_0 \left(\frac{\dot{x}}{\ell_1}\right)^2 + \frac{1}{2}\left(\frac{2}{5} m_s r_s^2\right) \dot{x}^2 \left(\frac{\ell_2}{\ell_1 r_s}\right)^2 + \frac{1}{2} m_s \left(\frac{\dot{x}\, \ell_2}{\ell_1}\right)^2$$

since for a sphere, $J_s = \frac{2}{5}\, m_s\, r_s^2$. Equating this to $T = \frac{1}{2}\, m_{eq}\, \dot{x}^2$, we obtain

$$m_{eq} = m + J_0\, \frac{1}{\ell_1^2} + \frac{7}{5}\, m_s\, \frac{\ell_2^2}{\ell_1^2}$$

1.55 (a)



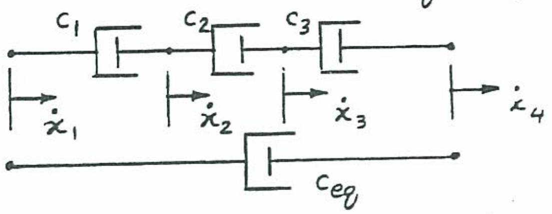
F_i = damping force of $c_i = c_i(\dot{x}_2 - \dot{x}_1)$; $i = 1,2,3$

F_{eq} = damping force of $c_{eq} = c_{eq}(\dot{x}_2 - \dot{x}_1)$

$\equiv F_1 + F_2 + F_3$

$\therefore \quad c_{eq} = c_1 + c_2 + c_3$

(b)



$F_1 = c_1(\dot{x}_2 - \dot{x}_1)$

$F_2 = c_2(\dot{x}_3 - \dot{x}_2)$

$F_3 = c_3(\dot{x}_4 - \dot{x}_3)$

$$\dot{x}_4 - \dot{x}_1 = \dot{x}_4 - \dot{x}_3 + \dot{x}_3 - \dot{x}_2 + \dot{x}_2 - \dot{x}_1$$

$$\frac{F_{eq}}{c_{eq}} = \frac{F_3}{c_3} + \frac{F_2}{c_2} + \frac{F_1}{c_1}$$

since $F_{eq} = F_1 = F_2 = F_3$, $\quad \frac{1}{c_{eq}} = \frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3}$

(c) Equating the energies dissipated in a cycle,

$$\pi\, c_{eq}\, \omega\, X_1^2 = \pi c_1 \omega\, X_1^2 + \pi\, c_2\, \omega\, X_2^2 + \pi\, c_3\, \omega\, X_3^2$$

where $X_1 = \theta l_1$, $X_2 = \theta l_2$ and $X_3 = \theta l_3$

$$\therefore \quad c_{eq} = c_1 + c_2\left(\frac{l_2}{l_1}\right)^2 + c_3\left(\frac{l_3}{l_1}\right)^2$$

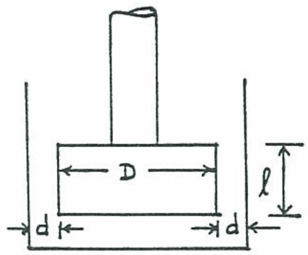
(d) Equating the energies dissipated in a cycle,

$$\pi\, c_{t\,eq}\, \omega\, \theta_1^2 = \pi\, c_{t1}\, \omega\, \theta_1^2 + \pi\, c_{t2}\, \omega\, \theta_2^2 + \pi\, c_{t3}\, \omega\, \theta_3^2$$

where $\theta_2 = \theta_1\left(\frac{n_1}{n_2}\right)$ and $\theta_3 = \theta_1\left(\frac{n_1}{n_3}\right)$.

$$\therefore \quad c_{t\,eq} = c_{t1} + c_{t2}\left(\frac{n_1}{n_2}\right)^2 + c_{t3}\left(\frac{n_1}{n_3}\right)^2.$$

1.57 Damping constant desired = c = 1 lb-sec/in, viscosity of the fluid = $\mu = 4\ \mu$ reyn = $4\ (10^{-6})$ lb—sec/in^2.



$$c = \mu \left\{ \frac{3\ \pi\ D^3\ \ell\ (1 + \frac{2\ d}{D})}{4\ d^3} \right\} \tag{1}$$

Assuming x = D/d as the unknown with $\ell = 2$ in, Eq. (1) can be written as

$$c = \mu \left(\frac{3\ \pi\ \ell\ x^3}{4} \right) (1 + \frac{2}{x}) \quad \text{or} \quad 1 = (4\ (10^{-6}))\ (\frac{3\ \pi\ (2)}{4})\ x^3\ (1 + \frac{2}{x}) \tag{2}$$

This gives $x^3 + 2\ x^2 - 53{,}051.52 = 0$

Using a trial and error procedure, the solution of this cubic equation can be found as $x \approx 36.92$. Using D = 3 in, we get d = 3/36.92 = 0.08126 in.

1.58 $c = \mu \left\{ \frac{3\pi D^3 \ell}{4 d^3} \left(1 + 2\frac{d}{D}\right) \right\}$;

D = diameter of piston
ℓ = axial length of piston
d = radial clearance

$\mu = 45$ μ reynolds
(from Shigley's Mechanical Engineering Design)

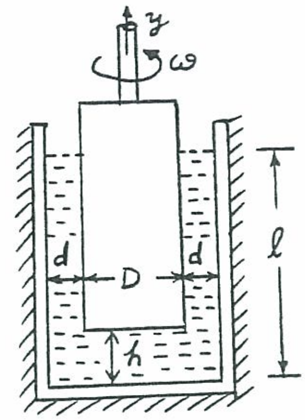
Let $d = 0.001''$, $D = 2.4''$ and above equation gives

$$10^5 = (45 \times 10^{-6}) \left\{ \frac{3\pi (2.4)^3 \ell}{4 (0.001)^3} \left(1 + \frac{2 \times 0.001}{2.4}\right) \right\}$$

$\therefore \ \ell = 0.6817''$

1.59 Tangential velocity of inner cylinder $= \frac{D}{2}\omega$

For small d, rate of change of velocity of fluid is $\frac{dv}{dr} = \frac{\frac{D}{2}\omega}{d}$



shear stress between cylinders is

$$\tau = \mu \frac{dv}{dr} = \mu \frac{D\omega}{2d}$$

and shear force is

$$F = \tau \cdot \text{Area} = \tau\, \pi D(l-h) = \frac{\pi\mu D^2\omega\,(l-h)}{2d}$$

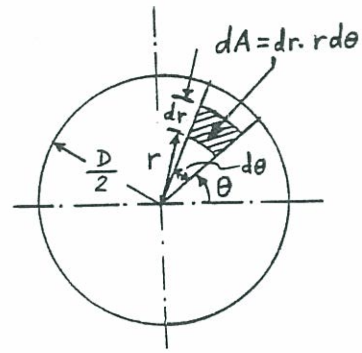
Torque developed $= M_{t1} = F \cdot \frac{D}{2}$

For small h, rate of change of velocity of fluid in vertical direction is

$$\frac{dv}{dy} = \frac{r\omega}{h}$$

Shear stress is $\tau = \mu \frac{dv}{dy} = \frac{\mu r \omega}{h}$

Force on area $dA = dF = \tau\, dA$



Torque between bottom surfaces of cylinders is

$$M_{t2} = \iint_{\text{area}} dM_{t2} \cdot dA \quad \text{where } dM_{t2} = dF \cdot r = \frac{\mu r^3 \omega}{h}\, dr\, d\theta$$

i.e., $$M_{t2} = \frac{\mu\omega}{h} \int_{r=0}^{D/2} \int_{\theta=0}^{2\pi} r^3 . dr\, d\theta = \frac{\mu\omega\pi D^4}{64\,h}$$

$$\text{Total torque} = M_t = M_{t1} + M_{t2} = \frac{\pi\mu D^3\omega\,(l-h)}{4d} + \frac{\pi\mu\omega D^4}{64\,h}$$

Expressing M_t as $c_t\, v = c_t\, \omega D/2$, we get damping constant:

$$c_t = \frac{\pi\mu D^2\,(l-h)}{2d} + \frac{\pi\mu D^3}{32\,h}$$

1.69

$$F = a\,\dot{x} + b\,\dot{x}^2 = 5\,\dot{x} + 0.2\,\dot{x}^2$$

$$F(\dot{x}) \approx F(\dot{x}_0) + \frac{dF}{d\dot{x}}\Big|_{\dot{x}_0} (\dot{x} - \dot{x}_0)$$

At $\dot{x}_0 = 5$ m/s, $F(\dot{x}_0) = 5\,(5) + 0.2\,(25) = 30$ N , $\frac{dF}{d\dot{x}}\Big|_{\dot{x}_0} = (5 + 0.4\,\dot{x})\Big|_5 = 7$ and hence

$F(\dot{x}) = 30 + 7\,(\dot{x} - 5) = 7\,\dot{x} - 5$.

Thus the linearized damping constant is given by $F(\dot{x}) \approx 7\,\dot{x} = c_{eq}\,\dot{x}$ or $c_{eq} = 7$ N–s/m.

1.70

Damping constant due to skin friction drag is:

$$c = 100\,\mu\,\ell^2\,d \tag{1}$$

Damping constant of a plate-type damper is:

$$c_p = \frac{\mu\,A}{h} \tag{2}$$

where A = area of plates and h = distance between the plates. If the area of plates (A) in Fig. 1.42 is taken to be same as the area of the plate shown in Fig. 1.107, we have $A = \ell\,d$. Equating (1) and (2) gives

$$100\,\mu\,\ell^2\,d = \frac{\mu\,\ell\,d}{h} \tag{3}$$

from which the clearance between the plates can be determined as $h = \frac{1}{100\,\ell}$.

1.71

$$c = \frac{6\,\pi\,\mu\,\ell}{h^3}\left\{(a - \frac{h}{2})^2 - r^2\right\}\left(\frac{a^2 - r^2}{a - \frac{h}{2}} - h\right)$$

When $\mu = 0.3445$ Pa-s, $\ell = 0.1$ m, $h = 0.001$ m, $a = 0..02$ m, and $r = 0.005$ m:

$$c = \frac{6\,\pi\,(0.3445)\,(0.1)}{(10^{-3})^3}\left\{(0.02 - 0.0005)^2 - 0.005^2\right\}\left(\frac{0.02^2 - 0.005^2}{0.02 - 0.0005} - 0.001\right)$$
$$= 4{,}205.6394 \text{ N–s/m}$$

1.72

$$c = \frac{6\pi\mu l}{h^3}\left[\left(a - \frac{h}{2}\right)^2 - r^2\right]\left[\frac{a^2 - r^2}{a - \frac{h}{2}} - h\right]$$

Basic data: $l = 10$ cm, $h = 0.1$ cm, $a = 2$ cm, $r = 0.5$ cm, $\mu = 0.3445$

Damping constant with basic data:

$c = 4,205.6230$ N-s/m

(a) r changed to 1 cm; new $c = 2,617.7920$ N-s/m

(b) h changed to 0.05 cm; new $c = 35,060.8910$ N-s/m

(c) a changed to 4 cm; new $c = 38,754.5860$ N-s/m

1.75 $\vec{x} = 5 + 2i = Ae^{i\theta} = A\cos\theta + iA\sin\theta$

$A\cos\theta = 5$

$A\sin\theta = 2$

$$A = \sqrt{(A\cos\theta)^2 + (A\sin\theta)^2} = \sqrt{5^2 + 2^2} = 5.3852$$

$$\theta = \tan^{-1}\left(\frac{A\sin\theta}{A\cos\theta}\right) = \tan^{-1}\left(\frac{2}{5}\right) = 21.8014^\circ$$

1.76 $\vec{x}_1 = 1 + 2i = a_1 + a_2 i$, $\vec{x}_2 = 3 - 4i = b_1 + b_2 i$

$$\vec{x} = \vec{x}_1 + \vec{x}_2 = (a_1 + b_1) + i(a_2 + b_2) = 4 - 2i$$

$$= Ae^{i\theta} = A\cos\theta + iA\sin\theta$$

$$A = \sqrt{4^2 + (-2)^2} = 4.4721$$

$$\theta = \tan^{-1}\left(\frac{-2}{4}\right) = -26.5651^\circ$$

1.77 $z_1 = (3 - 4i)$, $z_2 = (1 + 2i)$

$z = z_1 - z_2 = (3 - 4i) - (1 + 2i) = 2 - 6i = A e^{i\theta}$

where $A = \sqrt{2^2 + (-6)^2} = 6.3246$ and $\theta = \tan^{-1}\left(\frac{-6}{2}\right) = \tan^{-1}(-3) = -1.2490$ rad

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1.78 $z_1 = 1 + 2i$, $z_2 = 3 - 4i$

$z = z_1 z_2 = (1 + 2i)(3 - 4i) = 11 + 2i = A e^{i\theta}$

where $A = \sqrt{11^2 + 2^2} = 11.1803$ and $\theta = \tan^{-1}(2/11) = 0.1798$ rad

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1.79 $z = \dfrac{z_1}{z_2} = \dfrac{1 + 2i}{3 - 4i} = \dfrac{(1 + 2i)(3 + 4i)}{(3 - 4i)(3 + 4i)} = \dfrac{-5 + 10i}{25} = -0.2 + 0.4i = A e^{i\theta}$

where $A = \sqrt{(-0.2)^2 + (0.4)^2} = 0.4472$

and $\theta = \tan^{-1}\left(\frac{-0.4}{0.2}\right) = \tan^{-1}(-2) = -1.1071$ rad

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1.80 $x(t) = X \cos \omega t\,, \quad y(t) = Y \cos(\omega t + \phi)$

(a) $\dfrac{x^2}{X^2} = \cos^2 \omega t, \; \dfrac{y^2}{Y^2} = \cos^2(\omega t + \phi),$

$2\,\dfrac{x\,y}{X\,Y} \cos \phi = 2 \cos \omega t \cos(\omega t + \phi) \cos \phi$

$$\frac{x^2}{X^2} + \frac{y^2}{Y^2} - 2\,\frac{x\,y}{X\,Y} \cos \phi$$

$$= \cos^2 \omega t + \cos^2(\omega t + \phi) - 2 \cos \omega t \cos \phi \cos(\omega t + \phi) \tag{1}$$

Noting that $\cos^2 \alpha = \dfrac{1}{2}(1 + \cos 2\alpha)$, Eq. (1) can be rewritten as

$$\frac{x^2}{X^2} + \frac{y^2}{Y^2} - 2\,\frac{x\,y}{X\,Y} \cos \phi$$

$$= \frac{1}{2} + \frac{1}{2} \cos 2\omega t + \frac{1}{2} + \frac{1}{2} \cos(2\omega t + 2\phi) - 2 \cos \omega t \cos \phi \cos(\omega t + \phi)$$

$$= 1 + \frac{1}{2}\left\{2 \cos \frac{2\omega t + 2\omega t + 2\phi}{2} \cos \frac{2\omega t - 2\omega t - 2\phi}{2}\right\}$$
$$- 2 \cos \omega t \cos \phi \cos(\omega t + \phi)$$

$$= 1 + \cos(2\omega t + \phi) \cos \phi - 2 \cos \omega t \cos \phi \cos(\omega t + \phi)$$

$$= 1 + \cos(2\omega t + \phi) \cos \phi - 2 \cos \phi \left\{\frac{1}{2}\left[\cos(\omega t + \phi - \omega t) + \cos(\omega t + \phi + \omega t)\right]\right\}$$

$$= 1 + \cos \phi \cos(2\omega t + \phi) - \cos \phi \left\{\cos \phi + \cos(2\omega t + \phi)\right\}$$

$$= 1 - \cos^2 \phi = \sin^2 \phi \tag{2}$$

(b) When $\phi = 0$, Eq. (2) reduces to

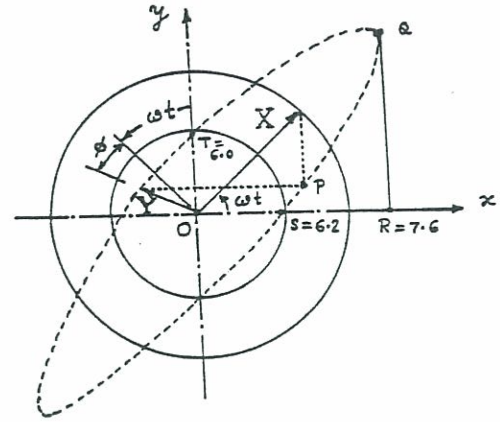
$$\frac{x^2}{X^2} + \frac{y^2}{Y^2} - 2\,\frac{x\,y}{X\,Y} = \left(\frac{x}{X} - \frac{y}{Y}\right)^2 = 0$$

which gives $X = \pm \dfrac{X}{Y}\,y$. This indicates that the locus of the resultant motion is a straight line. When $\phi = \dfrac{\pi}{2}$, Eq. (2) reduces to

$$\frac{x^2}{X^2} + \frac{y^2}{Y^2} = 1$$

which denotes an ellipse with its major and minor axes along x and y directions, respectively. When $\phi = \pi$, Eq. (2) reduces to that of a straight line as in the case of $\phi = 0$.

1.81



Equation for resultant motion:

$$\frac{x^2}{X^2} + \frac{y^2}{Y^2} - 2\,\frac{x\,y}{X\,Y}\cos^2\phi = \sin^2\phi \qquad (1)$$

When $y = 0$, Eq. (1) reduces to $\dfrac{x^2}{X^2} = \sin^2\phi$ and hence:

$$x = \pm X \sin\phi = \pm\, 6.2 = \text{OS in figure} \qquad (2)$$

When $x = 0$, Eq. (1) reduces to $\dfrac{y^2}{Y^2} = \sin^2\phi$ and hence:

$$y = \pm Y \sin\phi = \pm\, 6.0 = \text{OT in figure} \qquad (3)$$

It can be seen that

$$\text{OR} = X\cos\phi = 7.6 \text{ in figure} \qquad (4)$$

$$\frac{\text{OS}}{\text{OR}} = \frac{X\sin\phi}{X\cos\phi} = \tan\phi = \frac{6.2}{7.6} = 0.8158 \quad \text{or} \quad \phi = 39.2072^\circ \qquad (5)$$

From Eqs. (2) and (4), we find

$$X = \sqrt{(X\sin\phi)^2 + (X\cos\phi)^2} = \sqrt{(6.2)^2 + (7.6)^2} = 9.8082 \text{ mm}$$

Equations (3) and (5) give

$$Y = \frac{6.0}{\sin\phi} = \frac{6.0}{\sin 39.2072^\circ} = 9.4918 \text{ mm}$$

1.82 (a) $x(t) = \frac{A}{1000}\cos(50t+\alpha)$ m where A is in mm ---- (E_1)

$x(0) = \frac{A}{1000}\cos\alpha = 0.003$, $A\cos\alpha = 3$ ---- (E_2)

$\dot{x}(0) = -\frac{50A}{1000}\sin\alpha = 1$, $A\sin\alpha = -20$ ---- (E_3)

$$A = \{(A\cos\alpha)^2 + (A\sin\alpha)^2\}^{1/2} = 20.2237 \text{ mm}$$

$$\alpha = \tan^{-1}\left(\frac{A\sin\alpha}{A\cos\alpha}\right) = \tan^{-1}(-6.6667) = -81.4692^\circ = -1.4219 \text{ rad}$$

$$x(t) = 20.2237\cos(50t - 1.4219) \text{ mm}$$

(b) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

Eq. (E_1) can be expressed as $x(t) = A\cos 50t\cdot\cos\alpha - A\sin 50t\cdot\sin\alpha$

$= A_1\cos\omega t + A_2\sin\omega t$

where $\omega = 50$, $A_1 = A\cos\alpha$, $A_2 = -A\sin\alpha$

$\therefore\ x(t) = (3\cos 50t + 20\sin 50t)$ mm

1.83 $x(t) = A_1\cos\omega t + A_2\sin\omega t$

$\frac{dx}{dt}(t) = -A_1\omega\sin\omega t + A_2\omega\cos\omega t$, $\frac{d^2x}{dt^2} = -A_1\omega^2\cos\omega t - A_2\omega^2\sin\omega t$

$\frac{d^2x}{dt^2} = -\omega^2 x(t)$ where ω^2 is a constant

Hence $x(t)$ is a simple harmonic motion.

1.84 (a) Using trigonometric relations:

$x_1(t) = 5(\cos 3t\cos 1 - \sin 3t\sin 1)$

$x_2(t) = 10(\cos 3t\cos 2 - \sin 3t\sin 2)$

$x(t) = x_1(t) + x_2(t) = \cos 3t(5\cos 1 + 10\cos 2) - \sin 3t(5\sin 1 + 10\sin 2)$

If $x(t) = A\cos(\omega t+\alpha) = A\cos\omega t\cos\alpha - A\sin\omega t\sin\alpha$,

$\omega = 3$, $A\cos\alpha = 5\cos 1 + 10\cos 2 = -1.4599$,

$A\sin\alpha = 5\sin 1 + 10\sin 2 = 13.3003$

$$A = \sqrt{(A\cos\alpha)^2 + (A\sin\alpha)^2} = 13.3802$$

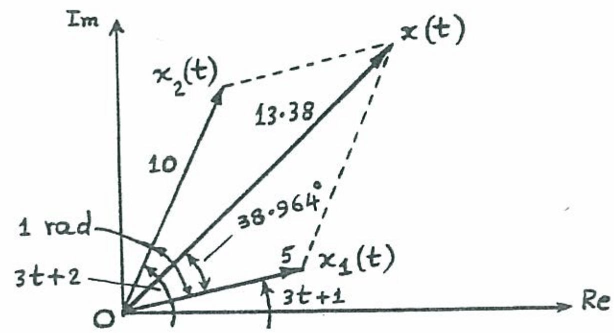
$$\alpha = \tan^{-1}\left(\frac{A\sin\alpha}{A\cos\alpha}\right) = \tan^{-1}(-9.1104) = 96.2640^\circ = 1.68 \text{ rad}$$

Angle between $x_1(t)$ and $x(t)$ is $96.2640^\circ - 57.3^\circ = 38.964^\circ$

(b) Using vector addition:

For an arbitrary value of $(\omega t+1)$, harmonic motions $x_1(t)$ and $x_2(t)$ can be shown as in the figure. From vector addition, we find

$x(t) \simeq 13.38\cos(\omega t + 1.68)$



(c) Using complex numbers:

$$x_1(t) = \text{Re}\left\{A_1\ e^{i(\omega t+1)}\right\} = \text{Re}\left\{5\ e^{i(\omega t+1)}\right\}$$

$$x_2(t) = \text{Re}\left\{A_2\ e^{i(\omega t+2)}\right\} = \text{Re}\left\{10\ e^{i(\omega t+2)}\right\}$$

If $x(t) = \text{Re}\left\{A\ e^{i(\omega t+\alpha)}\right\}$,

$$A\cos(3t+\alpha) = A_1\cos(3t+1) + A_2\cos(3t+2)$$

i.e. $$A(\cos 3t \cos\alpha - \sin 3t \sin\alpha) = 5(\cos 3t \cdot \cos 1 - \sin 3t \cdot \sin 1) + 10(\cos 3t \cdot \cos 2 - \sin 3t \cdot \sin 2)$$

i.e. $A\cos\alpha = 5\cos 1 + 10\cos 2$, $\quad A\sin\alpha = 5\sin 1 + 10\sin 2$

$A = 13.3802$, $\alpha = 1.68$ rad

$$x(t) = \text{Re}\left\{13.3802\ e^{i(3t+1.68)}\right\}$$

(1.85) $x(t) = 10\sin(\omega t + 60^\circ) = x_1(t) + x_2(t)$

where $x_1(t) = 5\sin(\omega t + 30^\circ)$ and $x_2(t) = A\sin(\omega t + \alpha^\circ)$

$$10(\sin\omega t\cos 60^\circ + \cos\omega t\sin 60^\circ) = 5(\sin\omega t\cdot\cos 30^\circ + \cos\omega t\cdot\sin 30^\circ) + A(\sin\omega t\cdot\cos\alpha^\circ + \cos\omega t\cdot\sin\alpha^\circ)$$

$10\cos 60^\circ = 5\cos 30^\circ + A\cos\alpha^\circ$; $\quad A\cos\alpha^\circ = 0.6699$

$10\sin 60^\circ = 5\sin 30^\circ + A\sin\alpha^\circ$; $\quad A\sin\alpha^\circ = 6.1603$

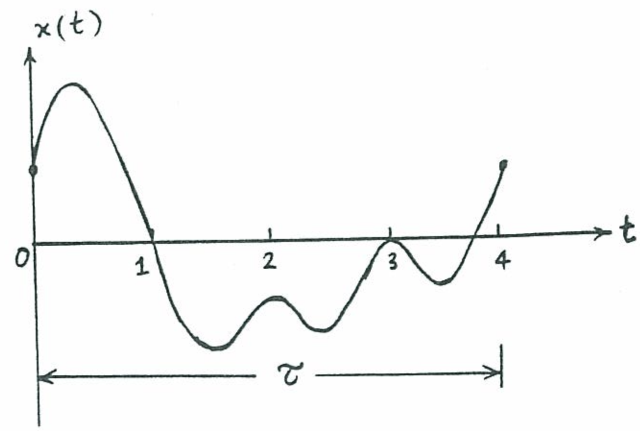
$$A = \sqrt{0.6699^2 + 6.1603^2} = 6.1966$$

$$\alpha = \tan^{-1}(6.1603/0.6699) = 83.7938^\circ$$

$$x_2(t) = 6.1966\sin(\omega t + 83.7938^\circ)$$

(1.86) $x(t) = \frac{1}{2}\cos\frac{\pi}{2}t + \sin\pi t$

$$= \frac{1}{2}\cos\frac{\pi}{2}t\left(1 + 4\sin\frac{\pi}{2}t\right)$$



From the nature of the graph of $x(t)$, it can be seen that $x(t)$ is periodic with a time period of $\tau = 4$.

(1.87) If $x(t)$ is harmonic, $\ddot{x}(t) = -\omega^2 x(t)$

Here $x(t) = 2\cos 2t + \cos 3t$

$\ddot{x}(t) = -8\cos 2t - 9\cos 3t \neq -$ constant times $x(t)$

$\therefore$ $x(t)$ is not harmonic

1.88 $x(t) = \frac{1}{2} \cos \frac{\pi}{2} t - \cos \pi t$

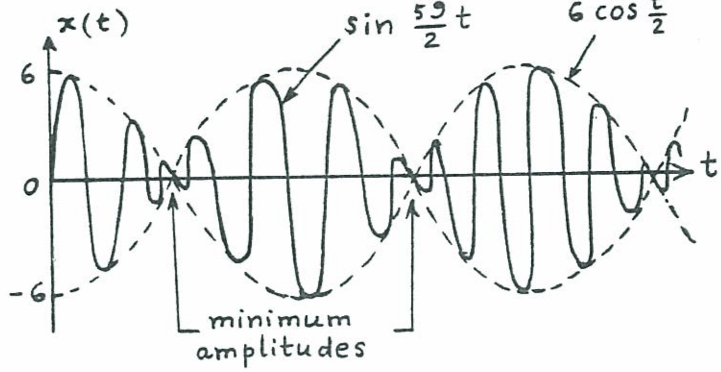
$\ddot{x}(t) = -\frac{\pi^2}{8} \cos \frac{\pi}{2} t + \pi^2 \cos \pi t \neq -$ constant times $x(t)$

$\therefore$ $x(t)$ is not harmonic

1.89 $x(t) = x_1(t) + x_2(t) = 3 \sin 30t + 3 \sin 29t$

since $\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$,

$x(t) = \left(6 \cos \frac{t}{2}\right) \sin \frac{59}{2} t$



This equation shows that the amplitude $(6 \cos \frac{t}{2})$ varies with time between a maximum value of 6 and a minimum value of 0. The frequency of this oscillation (beat frequency) is $\omega_b = 1$.

Note: Beat frequency is twice the frequency of the term $6 \cos \frac{t}{2}$ since two peaks pass in each cycle of $(6 \cos \frac{t}{2})$.

1.90 The resultant motion of two harmonic motions having identical amplitudes (X) but slightly different frequencies (ω and $\omega + \delta\omega$) is given by Eq. (1.67):

$$x(t) = 2\, X \cos \left(\omega\, t + \frac{\delta\omega\, t}{2} \right) \cos \left(\frac{\delta\, \omega\, t}{2} \right)$$

Thus the maximum amplitude of the resultant motion is equal to 2X and the beat frequency is equal to $\delta\omega$. From Fig. 1.113, we find that $2X \approx 5$ mm or $X = 2.5$ mm and

$$\frac{\delta\omega}{2} = \frac{2\,\pi}{\tau_{\text{beat}}} = \frac{2\,\pi}{\tau_{\text{larger}}} = \frac{2\,\pi}{2\,(12.6 - 4.2)} = 0.374 \text{ rad/sec}$$

or $\delta\omega = 0.748$ rad/sec and $\omega + \dfrac{\delta\,\omega}{2} = \dfrac{2\,\pi}{\tau_{\text{smaller}}} = \dfrac{2\,\pi}{1} = 6.2832$ rad/sec

Hence $\omega = 6.2832 - 0.3740 = 5.9092$ rad/sec. Thus the amplitudes of the two motions $= X = 2.5$ mm and their frequencies are $\omega = 5.9092$ rad/sec and $\omega + \delta\omega = 5.9092 + 0.7480 = 6.6572$ rad/sec.

1.91 $A = 0.05$ m, $\omega = 10$ Hz $= 62.832$ rad/sec

period $= \tau = \frac{2\pi}{\omega} = \frac{2\pi}{62.832} = 0.1$ sec

maximum velocity $= A\omega = 0.05 \times 62.832 = 3.1416$ m/s

maximum acceleration $= A\omega^2 = 0.05\,(62.832)^2 = 197.393$ m/s^2

1.92

$\omega = 15$ cps $= 94.248$ rad/sec

$\ddot{x}_{max} = 0.5 g = 0.5(9.81) = 4.905$ m/s^2 $= A\omega^2$

A = amplitude = $4.905/(94.248)^2 = 0.0005522$ m

$\dot{x}_{max}$ = max. velocity $= A\omega = 0.05204$ m/s

1.93

$x = A \cos \omega t$, $x_{max} = A = 0.25$ mm , $\ddot{x} = -\omega^2 A \cos \omega t$

$\ddot{x}_{max} = A\omega^2 = 0.4 g = 3924$ mm/s^2 ; $\omega^2 = 2924/A = 15696$ (rad/s)^2

operating speed of pump $= \omega = 125.2837$ rad/s $= 19.9395$ rpm

1.104

$$x(t) = X \sin \frac{2\pi t}{\tau} \; ; \; x_{rms} = \left[\frac{1}{\tau} \int_0^{\tau} X^2 \, \text{sun}^2 \, \frac{2\pi t}{\tau} \, dt \right]^{\frac{1}{2}}$$

Using $\sin^2 \frac{2\pi t}{\tau} = \dfrac{1 - \cos \frac{4\pi t}{\tau}}{2}$, we obtain

$$x_{rms} = \left[\frac{X^2}{\tau} \int_0^{\tau} \left(\frac{1}{2} - \frac{1}{2} \cos \frac{4\pi t}{\tau} \right) dt \right]^{\frac{1}{2}} = \left[\frac{X^2}{\tau} \left\{ \frac{t}{2} - \frac{1}{2} \, \frac{\tau}{4\pi} \sin \frac{4\pi t}{\tau} \right\} \Big|_0^{\tau} \right]^{\frac{1}{2}}$$

$$= \left[\frac{X^2}{\tau} \left\{ \frac{\tau}{2} - \frac{\tau}{8\pi} \sin 4\pi - 0 + 0 \right\} \right]^{\frac{1}{2}} = \frac{X}{\sqrt{2}}$$

1.105

$$x(t) = \frac{A\,t}{\tau} \; ; \; 0 \le t \le \tau$$

$$x_{rms} = \left\{ \frac{1}{\tau} \int_0^{\tau} \frac{A^2}{\tau^2} t^2 \, dt \right\}^{\frac{1}{2}} = \left\{ \frac{1}{\tau} \, \frac{A^2}{\tau^2} \left(\frac{t^3}{3} \right)_0^{\tau} \right\}^{\frac{1}{2}} = \left\{ \frac{A^2}{\tau^3} \, \frac{\tau^3}{3} \right\}^{\frac{1}{2}} = \left(\frac{A^2}{3} \right)^{\frac{1}{2}} = \frac{A}{\sqrt{3}}$$

1.106

For even functions, $x(-t) = x(t)$.

From Eq. (1.73), $b_n = \frac{2}{\tau} \int_0^{\tau} x(t) \sin n\omega t \cdot dt = \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} x(t) \sin n\omega t \; dt$

$$= \frac{2}{\tau} \left[\int_{-\frac{\tau}{2}}^{0} x(t) \sin n\omega t \cdot dt + \int_0^{\tau/2} x(t) \sin n\omega t \cdot dt \right] \quad \text{---- } (E_1)$$

Since $\sin(-n\omega t) = -\sin(n\omega t) =$ odd function of t, the product of $x(t)$ and $\sin n\omega t$ is an odd function.

Further, for an odd function $f(t)$, $f(-t) = -f(t)$, and

$$\int_{-a}^{a} f(t)\,dt = \int_{-a}^{0} f(t)\,dt + \int_{0}^{a} f(t)\,dt = \int_{0}^{a} f(-t)\,dt + \int_{0}^{a} f(t)\,dt$$

$$= -\int_{0}^{a} f(t)\,dt + \int_{0}^{a} f(t)\,dt = 0 \qquad \text{----}(E_2)$$

Equations (E_1) and (E_2) lead to $b_n = 0$.

Also, since $\cos n\omega t$ is an even function, we get

$$a_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} x(t)\cos n\omega t\,dt = \frac{4}{\tau}\int_{0}^{\tau/2} x(t)\cos n\omega t\,dt$$

<u>For odd functions,</u> $x(-t) = -x(t)$.

From Eq. (1.72), $a_n = \frac{2}{\tau}\int_{0}^{\tau} x(t)\cos n\omega t\,dt = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} x(t)\cos n\omega t\,dt$

Since $\cos n\omega t$ is an even function, $\cos(-n\omega t) = \cos(n\omega t)$, the product of $x(t)$ and $\cos n\omega t$ is an odd function.

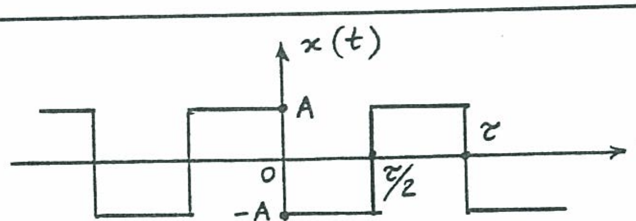
Hence $a_n = 0$.

Further, since $\sin n\omega t$ is an odd function, $x(t)\sin n\omega t$ is an even function and hence

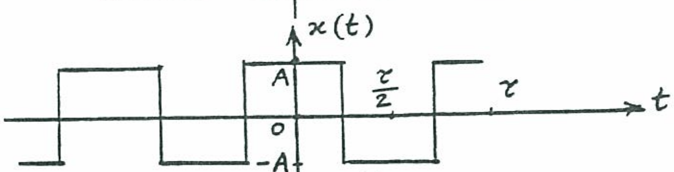
$$b_n = \frac{4}{\tau}\int_{0}^{\tau/2} x(t)\sin n\omega t\,dt$$

1.107

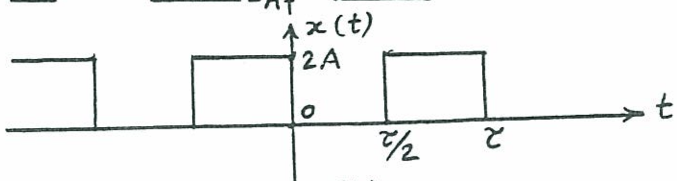
$$x(t) = \begin{cases} -A, & 0 \le t \le \frac{\tau}{2} \\ A, & \frac{\tau}{2} \le t \le \tau \end{cases} \qquad \text{(a)}$$



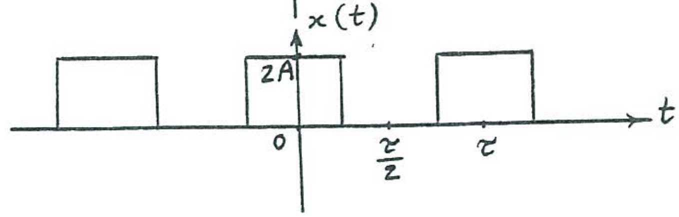
$$x(t) = \begin{cases} A, & 0 \le t \le \tau/4 \\ -A, & \frac{\tau}{4} \le t \le \frac{3\tau}{4} \\ A, & \frac{3\tau}{4} \le t \le \tau \end{cases} \qquad \text{(b)}$$



$$x(t) = \begin{cases} 0, & 0 \le t \le \tau/2 \\ 2A, & \frac{\tau}{2} \le t \le \tau \end{cases} \qquad \text{(c)}$$



$$x(t) = \begin{cases} 2A, & 0 \le t \le \frac{\tau}{4} \\ 0, & \frac{\tau}{4} \le t \le \frac{3\tau}{4} \\ 2A, & \frac{3\tau}{4} \le t \le \tau \end{cases} \qquad \text{(d)}$$



(a) $x(-t) = -x(t)$, odd function, hence $a_0 = a_n = 0$

$$b_n = \frac{2}{\tau}\int_0^\tau x(t)\sin n\omega t \cdot dt = \frac{2}{\tau}\left[-A\int_0^{\tau/2}\sin n\omega t \cdot dt + A\int_{\tau/2}^{\tau}\sin n\omega t \cdot dt\right]$$

$$= -\frac{2A}{\tau}\left(-\frac{\cos n\omega t}{n\omega}\right)_0^{\pi/\omega} + \frac{2A}{\tau}\left(-\frac{\cos n\omega t}{n\omega}\right)_{\pi/\omega}^{2\pi/\omega}$$

$$= \frac{2A}{\tau n\omega}\left(2\cos n\pi - \cos 0 - \cos 2n\pi\right)$$

$$\therefore x(t) = \frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin(2n-1)\omega t$$

(b) $x(-t) = x(t)$, even function, hence $b_n = 0$

$$a_0 = \frac{2}{\tau}\int_0^\tau x(t)\,dt = \frac{2}{\tau}\left[A\cdot(t)_0^{\tau/4} - A(t)_{\tau/4}^{3\tau/4} + A(t)_{3\tau/4}^{\tau}\right] = 0$$

$$a_n = \frac{2}{\tau}\int_0^\tau x(t)\cos n\omega t \cdot dt$$

$$= \frac{2A}{\tau n\omega}\left[\sin n\omega t\Big|_0^{\tau/4} - \sin n\omega t\Big|_{\tau/4}^{\frac{3\tau}{4}} + \sin n\omega t\Big|_{3\tau/4}^{\tau}\right]$$

$$= \frac{A}{n\pi}\left[2\sin\frac{n\pi}{2} - 2\sin\frac{3n\pi}{2} + \sin 2\pi n\right] = \begin{cases} 4A/n\pi & \text{for } n = 1, 5, 9, \ldots \\ -4A/n\pi & \text{for } n = 3, 7, 11, \ldots \end{cases}$$

$$\therefore x(t) = \frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)}\cos\frac{2\pi(n-1)t}{\tau}$$

(c) $$a_0 = \frac{2}{\tau}\int_0^\tau x(t)\,dt = \frac{2}{\tau}\left[0 + 2A\;(t)\Big|_{\tau/2}^{\tau}\right] = 2A$$

$$a_n = \frac{2}{\tau}\int_0^\tau x(t)\cos n\omega t\,dt = \frac{4A}{n\omega\tau}\left(\sin n\omega t\right)_{\tau/2}^{\tau} = 0$$

$$b_n = \frac{2}{\tau}\int_0^\tau x(t)\sin n\omega t\,dt = -\frac{4A}{n\omega\tau}\left(\cos n\omega t\right)_{\tau/2}^{\tau}$$

$$= -\frac{4A}{n\omega\tau}\left(\cos 2\pi n - \cos n\pi\right)$$

$$\therefore x(t) = -\frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin(2n-1)\omega t \qquad \text{with } \omega = 2\pi/\tau.$$

(d) $x(-t) = x(t)$, even function, hence $b_n = 0$

$$a_0 = \frac{2}{\tau}\int_0^\tau x(t)\,dt = \frac{2}{\tau}\left[2A\left(\frac{\tau}{4} - 0\right) + 2A\left(\tau - \frac{3\tau}{4}\right)\right] = 2A$$

$$a_n = \frac{2}{\tau}\int_0^\tau x(t)\cos n\omega t\,dt = \frac{4A}{n\omega\tau}\left[\left(\sin n\omega t\right)_0^{\tau/4} + \left(\sin n\omega t\right)_{\frac{3\tau}{4}}^{\tau}\right]$$

$$= \frac{4A}{n\omega\tau}\left(\sin\frac{n\pi}{2} + \sin 2n\pi - \sin\frac{3n\pi}{2}\right)$$

$$\therefore\ x(t) = \frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)}\cos\frac{2\pi(2n-1)t}{\tau} \quad \text{with } \omega = 2\pi/\tau.$$

1.108

$$x(t) = \begin{cases} A\sin\frac{2\pi t}{\tau}, & 0 \le t \le \frac{\tau}{2} \\ 0, & \frac{\tau}{2} \le t \le \tau \end{cases}$$

$$a_0 = \frac{2}{\tau}\int_0^{\tau} x(t)\,dt = \frac{2A}{\tau}\int_0^{\tau/2}\sin\frac{2\pi t}{\tau}\,dt = \frac{2A}{\tau}\left(-\frac{\tau}{2\pi}\cdot\cos\frac{2\pi t}{\tau}\right)_0^{\tau/2}$$

$$= \frac{2A}{\pi}$$

$$a_n = \frac{2}{\tau}\int_0^{\tau} x(t)\cos n\omega t\,dt = \frac{2A}{\tau}\int_0^{\tau/2}\sin\frac{2\pi t}{\tau}\cdot\cos n\omega t\cdot dt \quad ---\ (E_1)$$

Using the relation $\sin m\omega t\cdot\cos n\omega t = \frac{\sin(m+n)\omega t + \sin(m-n)\omega t}{2}$, Eq. (E_1) can be rewritten as

$$a_n = \frac{A}{\tau}\int_0^{\pi/\omega}\left[\sin(1+n)\omega t + \sin(1-n)\omega t\right]dt$$

when $n = 1$, $a_1 = \frac{A}{\tau}\int_0^{\pi/\omega}\sin 2\omega t\cdot dt = 0$

when $n = 2, 3, 4, \ldots$, $a_n = \frac{A}{\tau}\left[-\frac{\cos(1+n)\omega t}{(1+n)\omega} - \frac{\cos(1-n)\omega t}{(1-n)\omega}\right]_0^{\pi/\omega}$

$$= \frac{A}{2\pi}\left[\frac{1-\cos(1+n)\pi}{1+n} + \frac{1-\cos(1-n)\pi}{1-n}\right]$$

$$= \begin{cases} 0 & \text{if } n \text{ is odd} \\ -\frac{2A}{(n-1)(n+1)\pi} & \text{if } n \text{ is even} \end{cases}$$

Similarly

$$b_n = \frac{2}{\tau}\int_0^{\tau} x(t)\sin n\omega t\,dt = \frac{2A}{\tau}\int_0^{\tau/2}\sin\frac{2\pi t}{\tau}\cos n\omega t\,dt$$

$$= \frac{A}{\tau}\int_0^{\tau/2}\left[\cos(1-n)\omega t - \cos(1+n)\omega t\right]dt$$

when $n = 1$, $b_1 = \frac{A}{\tau}\int_0^{\pi/\omega}(dt - \cos 2\omega t)\,dt = \frac{A}{2}$

when $n = 2, 3, 4, \ldots$, $b_n = \frac{A}{\tau}\left[\frac{\sin(1-n)\omega t}{(1-n)\omega} - \frac{\sin(1+n)\omega t}{(1+n)\omega}\right]_0^{\pi/\omega} = 0$

$$\therefore\ x(t) = \frac{A}{\pi} + \frac{A}{2}\sin\omega t - \frac{2A}{\pi}\sum_{n=2,4,6,\ldots}\frac{\cos n\omega t}{(n^2-1)}$$

1.109

$$x(t) = \begin{cases} \frac{2At}{\tau}, & 0 \le t \le \frac{\tau}{2} \\ -\frac{2At}{\tau} + 2A, & \frac{\tau}{2} \le t \le \tau \end{cases}$$

$$a_0 = \frac{2}{\tau}\int_0^{\tau} x(t)\,dt = \frac{2}{\tau}\left[\int_0^{\tau/2} \frac{2At}{\tau}\cdot dt + \int_{\tau/2}^{\tau}\left(-\frac{2At}{\tau} + 2A\right)\cdot dt\right]$$

$$= \frac{2}{\tau}\left[\frac{2A}{\tau}\cdot\frac{t^2}{2}\Big|_0^{\tau/2} - \frac{2A}{\tau}\cdot\frac{t^2}{2}\Big|_{\tau/2}^{\tau} + 2A\cdot t\Big|_{\tau/2}^{\tau}\right]$$

$$= \frac{2}{\tau}\left[\frac{A\tau}{4} - \frac{3A\tau}{4} + A\tau\right] = A$$

$$a_n = \frac{2}{\tau}\int_0^{\tau} x(t)\cos n\omega t\,dt$$

$$= \frac{2}{\tau}\left[\int_0^{\frac{\tau}{2}} \frac{2A}{\tau}\,t\cos n\omega t\,dt + \int_{\tau/2}^{\tau}\left(-\frac{2A}{\tau}t + 2A\right)\cos n\omega t\,dt\right]$$

$$= \frac{2}{\tau}\left[\frac{2A}{\tau}\left\{t\,\frac{\sin n\omega t}{n\omega} + \frac{\cos n\omega t}{n^2\omega^2}\right\}_0^{\tau/2}\right.$$

$$\left. - \frac{2A}{\tau}\left\{t\,\frac{\sin n\omega t}{n\omega} + \frac{\cos n\omega t}{n^2\omega^2}\right\}_{\tau/2}^{\tau} + 2A\left(-\frac{\sin n\omega t}{n\omega}\right)_{\tau/2}^{\tau}\right]$$

As $\tau = \frac{2\pi}{\omega}$,

$$a_n = \frac{\omega}{\pi}\left[\frac{A\omega}{\pi n^2\omega^2}\cos n\pi - \frac{A\omega}{\pi n^2\omega^2} - \frac{A\omega}{\pi n^2\omega^2}\cos 2\pi n + \frac{A\omega}{\pi n^2\omega^2}\cos n\pi\right]$$

$$= \frac{2A}{n^2\pi^2}\left(\cos n\pi - 1\right) = \begin{cases} -\frac{4A}{n^2\pi^2}, & n = 1, 3, 5, \ldots \\ 0, & n = 2, 4, 6, \ldots \end{cases}$$

$$b_n = \frac{2}{\tau}\int_0^{\tau} x(t)\sin n\omega t\,dt = \frac{2}{\tau}\left[\int_0^{\tau/2}\frac{2A}{\tau}\,t\sin n\omega t\,dt\right.$$

$$\left. + \int_{\tau/2}^{\tau}\left(-\frac{2A}{\tau}t + 2A\right)\sin n\omega t\,dt\right]$$

$$= \frac{2}{\tau}\left[\frac{2A}{\tau}\left\{-\frac{t\cos n\omega t}{n\omega} + \frac{\sin n\omega t}{n^2\omega^2}\right\}_0^{\tau/2}\right.$$

$$\left. - \frac{2A}{\tau}\left\{-\frac{t\cos n\omega t}{n\omega} + \frac{\sin n\omega t}{n^2\omega^2}\right\}_{\tau/2}^{\tau} + 2A\left(-\frac{\cos n\omega t}{n\omega}\right)_{\tau/2}^{\tau}\right]$$

$$= \frac{\omega}{\pi}\left[-\frac{A}{n\omega}\cos n\pi + \frac{2A}{n\omega}\cos 2n\pi - \frac{A}{n\omega}\cos n\pi - \frac{2A}{n\omega}\cos 2n\pi + \frac{2A}{n\omega}\cos n\pi\right] = 0$$

$$\therefore \; x(t) = \frac{A}{2} - \frac{4A}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2} \cos n\omega t$$

1.110

$$x(t) = \begin{cases} \dfrac{4At}{\tau} & , \; 0 \le t \le \dfrac{\tau}{4} \\ -\dfrac{4At}{\tau} + 2A & , \; \dfrac{\tau}{4} \le t \le \dfrac{3\tau}{4} \\ \dfrac{4At}{\tau} - 4A & , \; \dfrac{3\tau}{4} \le t \le \tau \end{cases}$$

$$a_0 = \frac{2}{\tau}\int_0^{\tau} x(t)\, dt = \frac{2}{\tau}\left[\frac{4A}{\tau} \frac{t^2}{2}\Big|_0^{\tau/4} + \left(-\frac{4A}{\tau}\frac{t^2}{2} + 2At\right)_{\tau/4}^{3\tau/4} + \left(\frac{4A}{\tau}\frac{t^2}{2} - 4At\right)_{3\tau/4}^{\tau} \right] = 0$$

$$a_n = \frac{2}{\tau}\int_0^{\tau} x(t) \cos n\omega t \, dt = \frac{2}{\tau}\left[\frac{4A}{\tau}\int_0^{\tau/4} t \cos n\omega t \cdot dt - \frac{4A}{\tau}\int_{\tau/4}^{3\tau/4} t \cos n\omega t \, dt + 2A\int_{\tau/4}^{3\tau/4} \cos n\omega t \, dt + \frac{4A}{\tau}\int_{3\tau/4}^{\tau} t \cos n\omega t\, dt - 4A \int_{3\tau/4}^{\tau} \cos n\omega t \, dt \right]$$

$$= \frac{2}{\tau}\left[\frac{4A}{\tau}\left\{ t\frac{\sin n\omega t}{n\omega} + \frac{\cos n\omega t}{n^2\omega^2} \right\}_0^{\tau/4} - \frac{4A}{\tau}\left\{ t\frac{\sin n\omega t}{n\omega} + \frac{\cos n\omega t}{n^2\omega^2}\right\}_{\tau/4}^{3\tau/4} + 2A\left(\frac{\sin n\omega t}{n\omega}\right)_{\tau/4}^{3\tau/4} + \frac{4A}{\tau}\left\{ t\frac{\sin n\omega t}{n\omega} + \frac{\cos n\omega t}{n^2\omega^2}\right\}_{3\tau/4}^{\tau} - 4A\left(\frac{\sin n\omega t}{n\omega}\right)_{3\tau/4}^{\tau} \right]$$

$$= \frac{\omega}{\pi}\left[\sin\frac{n\pi}{2}\left(\frac{A}{n\omega} + \frac{A}{n\omega} - \frac{2A}{n\omega}\right) + \cos\frac{n\pi}{2}\left(\frac{2A}{\pi n^2\omega} + \frac{2A}{\pi n^2 \omega}\right) + \sin\frac{3n\pi}{2}\left(-\frac{3A}{n\omega} + \frac{2A}{n\omega} - \frac{3A}{n\omega} + \frac{4A}{n\omega}\right) + \cos\frac{3n\pi}{2}\left(-\frac{2A}{\pi n^2\omega} - \frac{2A}{\pi n^2\omega}\right) + \cos 2\pi n\left(\frac{2A}{\pi n^2\omega}\right) - \cos 0\left(\frac{2A}{\pi n^2\omega}\right) \right] = 0$$

$$b_n = \frac{2}{\tau}\int_0^{\tau} x(t) \sin n\omega t \, dt = \frac{2}{\tau}\left[\frac{4A}{\tau}\int_0^{\tau/4} t \sin n\omega t \cdot dt - \frac{4A}{\tau}\int_{\tau/4}^{3\tau/4} t \sin n\omega t \cdot dt + 2A\int_{\tau/4}^{3\tau/4} \sin n\omega t \cdot dt + \frac{4A}{\tau}\int_{3\tau/4}^{\tau} t \sin n\omega t \cdot dt - 4A\int_{3\tau/4}^{\tau} \sin n\omega t \cdot dt \right]$$

$$= \frac{2}{\tau}\left[\frac{4A}{\tau}\left\{ \frac{1}{n^2\omega^2}\sin n\omega t - \frac{t}{n\omega}\cos n\omega t \right\}_0^{\tau/4 = \frac{\pi}{2\omega}} - \frac{4A}{\tau}\left\{ \frac{1}{n^2\omega^2}\sin n\omega t - \frac{t}{n\omega}\cos n\omega t\right\}_{\frac{\tau}{4} = \frac{\pi}{2\omega}}^{\frac{3\tau}{4} = \frac{3\pi}{2\omega}} + 2A\left(-\frac{\cos n\omega t}{n\omega}\right)_{\frac{\tau}{4} = \frac{\pi}{2\omega}}^{\frac{3\tau}{4} = \frac{3\pi}{2\omega}} + \frac{4A}{\tau}\left\{ \frac{1}{n^2\omega^2} \sin n\omega t \right.$$

$$- \frac{t}{n\omega} \cos n\omega t \Big|_{\frac{3\tau}{4} = \frac{3\pi}{2\omega}}^{\tau = \frac{2\pi}{\omega}} - 4A \left(- \frac{\cos n\omega t}{n\omega} \right)_{\frac{3\tau}{4} = \frac{3\pi}{2\omega}}^{\tau = \frac{2\pi}{\omega}} \Big]$$

$$= \frac{4A}{\pi^2 n^2} \left(\sin \frac{n\pi}{2} - \sin \frac{3n\pi}{2} \right) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{8A}{\pi^2 n^2} (-1)^{\frac{n-1}{2}} & \text{if } n \text{ is odd} \end{cases}$$

$$\therefore\ x(t) = \frac{8A}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} (-1)^{\frac{n-1}{2}} \frac{\sin n\omega t}{n^2}$$

1.111

$$x(t) = A\left(1 - \frac{t}{\tau}\right), \quad 0 \le t \le \tau$$

$$a_0 = \frac{2}{\tau} \int_0^{\tau} x(t)\, dt = \frac{2A}{\tau} \int_0^{\tau} \left(1 - \frac{t}{\tau}\right) dt = \frac{2A}{\tau} \left(t - \frac{t^2}{2\tau}\right)_0^{\tau} = A$$

$$a_n = \frac{2}{\tau} \int_0^{\tau} x(t) \cos n\omega t\, dt = \frac{2A}{\tau} \left(\frac{\sin n\omega t}{n\omega} - \frac{t}{\tau} \frac{\sin n\omega t}{n\omega} - \frac{\cos n\omega t}{\tau n^2 \omega^2} \right)_0^{2\pi/\omega}$$

$$= 0$$

$$b_n = \frac{2}{\tau} \int_0^{\tau} x(t) \sin n\omega t\, dt = \frac{2A}{\tau} \left(- \frac{\cos n\omega t}{n\omega} + \frac{t}{\tau} \frac{\cos n\omega t}{n\omega} - \frac{\sin n\omega t}{\tau n^2 \omega^2} \right)_0^{2\pi/\omega}$$

$$= \frac{A}{\pi n}$$

$$\therefore\ x(t) = \frac{A}{2} + \frac{A}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\omega t}{n}$$

1.112

The truncated series of k terms can be denoted as

$$\overline{x}(t) = \frac{\overline{a}_0}{2} + \sum_{n=1}^{k} \overline{a}_n \cos n\,\omega\,t + \sum_{n=1}^{k} \overline{b}_n \sin n\,\omega\,t \tag{1}$$

with $\overline{x}(t)$ denoting an approximation to the exact x(t) given by Eq. (1.70). The error to be minimized is given by

$$E = \int_{-\pi/\omega}^{\pi/\omega} e^2(t)\, dt \tag{2}$$

$$\text{where} \quad e(t) = x(t) - \overline{x}(t) \tag{3}$$

and x(t) is the exact value (with infinite series on the right hand side of Eq. (1)). Treating E as a function of the unknowns $\overline{a}_n$ and $\overline{b}_n$, it can be minimized by setting:

$$\frac{\partial E}{\partial \overline{a}_n} = 2 \int_{-\pi/\omega}^{\pi/\omega} \left\{ x(t) - \overline{x}(t) \right\} \left(- \cos n\,\omega\,t \right) dt = 0 \tag{4}$$

$$\frac{\partial E}{\partial \overline{b}_n} = 2 \int_{-\pi/\omega}^{\pi/\omega} \left\{ x(t) - \overline{x}(t) \right\} \left(- \sin n\,\omega\,t \right) dt = 0 \tag{5}$$

Rearranging Eq. (4) gives

$$\int_{-\pi/\omega}^{\pi/\omega} x(t) \cos n\,\omega\, t\, dt = \int_{-\pi/\omega}^{\pi/\omega} \bar{x}(t) \cos n\,\omega\, t\, dt \tag{6}$$

Using orthogonalty property, the right hand side of Eq. (6) can be expressed as

$$\int_{-\pi/\omega}^{\pi/\omega} \bar{x}(t) \cos n\,\omega\, t\, dt = \begin{cases} 0 & \text{for } m \neq n \\ \dfrac{\bar{a}_n\,\pi}{\omega} & \text{for } m = n \end{cases} \tag{7}$$

This leads to

$$\int_{-\pi/\omega}^{\pi/\omega} x(t) \cos n\,\omega\, t\, dt = \frac{\bar{a}_n\,\pi}{\omega} \tag{8}$$

$$\text{or}\quad \bar{a}_n = \frac{\omega}{\pi} \int_{-\pi/\omega}^{\pi/\omega} x(t) \cos n\,\omega\, t\, dt \quad ; \quad n = 0,\ 1,\ 2,\ \ldots,\ k \tag{9}$$

In a similar manner, we can derive:

$$\bar{b}_n = \frac{\omega}{\pi} \int_{-\pi/\omega}^{\pi/\omega} x(t) \sin n\,\omega\, t\, dt \quad ; \quad n = 1,\ 2,\ \ldots,\ k \tag{10}$$

It can be observed that Eqs. (9) and (10) are simlar to those of Eqs. (E.3) and (E.4).

1.113

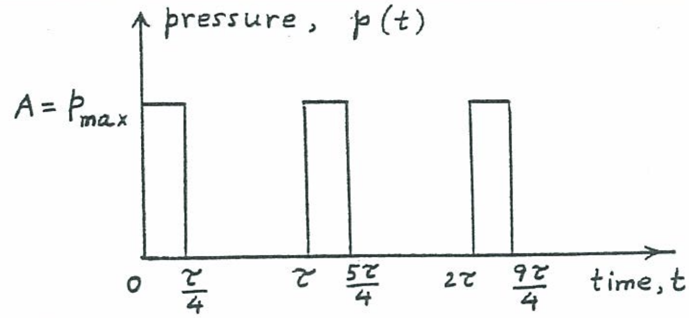
i	t_i	x_i	n = 1: $x_i \cos \frac{2\pi t_i}{0.32}$	n = 1: $x_i \sin \frac{2\pi t_i}{0.32}$	n = 2: $x_i \cos \frac{4\pi t_i}{0.32}$	n = 2: $x_i \sin \frac{4\pi t_i}{0.32}$	n = 3: $x_i \cos \frac{6\pi t_i}{0.32}$	n = 3: $x_i \sin \frac{6\pi t_i}{0.32}$
1	0.02	9	8.3149	3.4442	6.3639	6.3640	3.4441	8.3149
2	0.04	13	9.1924	9.1924	0.0000	13.0000	−9.1924	9.1923
3	0.06	17	6.5056	15.7060	−12.0209	12.0208	−15.7059	−6.5057
4	0.08	29	0.0000	29.0000	−29.0000	0.0000	0.0000	−29.0000
5	0.10	43	−16.4556	39.7267	−30.4053	−30.4059	39.7271	−16.4548
6	0.12	59	−41.7195	41.7191	0.0000	−59.0000	41.7187	41.7199
7	0.14	63	−58.2045	24.1087	44.5482	−44.5472	−24.1101	58.2040
8	0.16	57	−57.0000	0.0000	57.0000	0.0000	−57.0000	0.0000
9	0.18	49	−45.2700	−18.7518	34.6477	34.6487	−18.7505	−45.2705
10	0.20	35	−24.7485	−24.7489	0.0000	35.0000	24.7493	−24.7482
11	0.22	35	−13.3936	−32.3359	−24.7493	24.7482	32.3354	13.3950
12	0.24	41	0.0000	−41.0000	−41.0000	0.0000	0.0000	41.0000
13	0.26	47	17.9866	−43.4221	−33.2333	−33.2347	−43.4229	17.9847
14	0.28	41	28.9917	−28.9911	0.0000	−41.0000	−28.9905	−28.9923
15	0.30	13	12.0105	−4.9747	9.1927	−9.1921	4.9755	−12.0102
16	0.32	7	7.0000	0.0000	7.0000	0.0000	7.0000	0.0000

$\sum_{i=1}^{16}(\ )$	558	−166.7897	−31.3278	−11.6552	−91.5984	−43.2234	26.8281
$\frac{1}{8}\sum_{i=1}^{16}(\ )$	69.75	−20.8487	−3.9160	−1.4569	−11.4498	−5.4029	3.3535

1.114

Speed = 100 rpm

In a minute, a point will be subjected to the maximum pressure, $A = p_{max} = 100$ psi, $100 \times 4 = 400$ times. Hence



$$\text{period} = \tau = \frac{60}{400} = 0.15 \text{ sec.}$$

$$p(t) = \begin{cases} A, & 0 \le t \le \tau/4 \\ 0, & \tau/4 \le t \le \tau \end{cases}$$

$$a_0 = \frac{2}{\tau}\int_0^\tau p(t)\,dt = \frac{2}{\tau}A\left(t\right)_0^{\tau/4} = \frac{A}{2} = 50 \text{ psi}$$

$$a_m = \frac{2}{\tau}\int_0^\tau p(t)\cos m\omega t\,dt = \frac{2A}{\tau}\left(\frac{\sin m\omega t}{m\omega}\right)_0^{\tau/4} = \frac{A}{\pi m}\sin\frac{m\pi}{2}$$

$$b_m = \frac{2}{\tau}\int_0^\tau p(t)\sin m\omega t\,dt = -\frac{2A}{\tau}\left(\frac{\cos m\omega t}{m\omega}\right)_0^{\tau/4} = -\frac{A}{\pi m}\left(\cos\frac{m\pi}{2} - 1\right)$$

Evaluation of a_m and b_m:

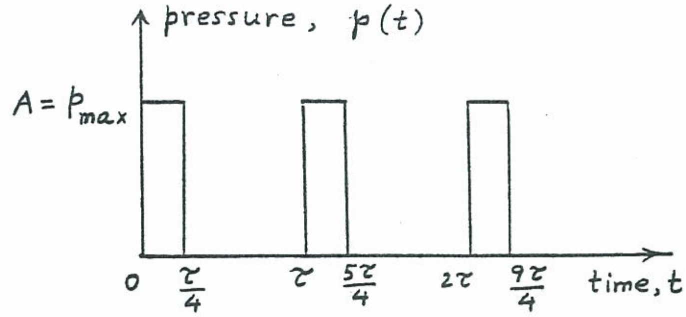
$m = 1$	$m = 2$	$m = 3$
$a_1 = \frac{A}{\pi}\sin\frac{\pi}{2} = \frac{A}{\pi}$ $= 31.8309$ psi	$a_2 = \frac{A}{2\pi}\sin\pi = 0$	$a_3 = \frac{A}{3\pi}\sin\frac{3\pi}{2}$ $= -10.6103$ psi
$b_1 = -\frac{A}{\pi}\left(\cos\frac{\pi}{2} - 1\right)$ $= 31.8309$ psi	$b_2 = -\frac{A}{2\pi}\left(\cos\pi - 1\right)$ $= 31.8309$ psi	$b_3 = -\frac{A}{3\pi}\left(\cos\frac{3\pi}{2} - 1\right)$ $= 10.6103$ psi

$$\therefore\ p(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m\cos m\omega t + b_m\sin m\omega t\right) \text{ psi}$$

1.115

Speed = 200 rpm
In a minute, a point will be subjected to the maximum pressure, $A = p_{max} = 100$ psi, $200 \times 6 = 1200$ times. Hence

period $= \tau = \frac{60}{1200} = 0.05$ sec.



$$p(t) = \begin{cases} A, & 0 \le t \le \tau/4 \\ 0, & \tau/4 \le t \le \tau \end{cases}$$

$$a_0 = \frac{2}{\tau}\int_0^\tau p(t)\,dt = \frac{2}{\tau} A \left(t\right)_0^{\tau/4} = \frac{A}{2} = 50 \text{ psi}$$

$$a_m = \frac{2}{\tau}\int_0^\tau p(t)\cos m\omega t\,dt = \frac{2A}{\tau}\left(\frac{\sin m\omega t}{m\omega}\right)_0^{\tau/4} = \frac{A}{\pi m}\sin\frac{m\pi}{2}$$

$$b_m = \frac{2}{\tau}\int_0^\tau p(t)\sin m\omega t\,dt = -\frac{2A}{\tau}\left(\frac{\cos m\omega t}{m\omega}\right)_0^{\tau/4} = -\frac{A}{\pi m}\left(\cos\frac{m\pi}{2} - 1\right)$$

Evaluation of a_m and b_m:

$m = 1$	$m = 2$	$m = 3$
$a_1 = \frac{A}{\pi}\sin\frac{\pi}{2} = \frac{A}{\pi} = 31.8309$ psi	$a_2 = \frac{A}{2\pi}\sin\pi = 0$	$a_3 = \frac{A}{3\pi}\sin\frac{3\pi}{2} = -10.6103$ psi
$b_1 = -\frac{A}{\pi}\left(\cos\frac{\pi}{2} - 1\right) = 31.8309$ psi	$b_2 = -\frac{A}{2\pi}\left(\cos\pi - 1\right) = 31.8309$ psi	$b_3 = -\frac{A}{3\pi}\left(\cos\frac{3\pi}{2} - 1\right) = 10.6103$ psi

$$\therefore\ p(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos m\omega t + b_m \sin m\omega t\right) \quad \text{psi}$$

1.116

			$n = 1$		$n = 2$		$n = 3$	
i	t_i	M_{t_i}	$M_{t_i}\cos\frac{2\pi t_i}{0.012}$	$M_{t_i}\sin\frac{2\pi t_i}{0.012}$	$M_{t_i}\cos\frac{4\pi t_i}{0.012}$	$M_{t_i}\sin\frac{4\pi t_i}{0.012}$	$M_{t_i}\cos\frac{6\pi t_i}{0.012}$	$M_{t_i}\sin\frac{6\pi t_i}{0.012}$
1	0.0005	770	743.7627	199.2912	666.8391	385.0010	544.4712	544.4731
2	0.0010	810	701.4802	405.0007	404.9988	701.4812	0.0000	810.0000
3	0.0015	850	601.0398	601.0417	0.0000	850.0000	−601.0442	601.0373
4	0.0020	910	454.9978	788.0845	−455.0041	788.0808	−910.0000	0.0000

5	0.0025	1010	261.4043	975.5859	−874.689	504.995	−714.171	−714.184
6	0.0030	1170	0.0000	1170.0000	−1170.000	0.000	0.000	−1170.000
7	0.0035	1370	−354.5874	1323.3169	−1186.449	−685.010	968.748	−968.725
8	0.0040	1610	−805.0073	1394.2966	−804.987	−1394.309	1610.000	0.000
9	0.0045	1890	−1336.4407	1336.4229	0.000	−1890.000	1336.410	1336.454
10	0.0050	1750	−1515.5491	874.9922	875.019	−1515.534	0.000	1750.000
11	0.0055	1630	−1574.4619	421.8647	1411.634	−814.979	−1152.608	1152.560
12	0.0060	1510	−1510.0000	0.0000	1510.000	0.000	−1510.000	0.000
13	0.0065	1390	−1342.6345	−359.7671	1203.767	695.014	−982.858	−982.898
14	0.0070	1290	−1117.1677	−645.0088	644.982	1117.183	0.000	−1290.000
15	0.0075	1190	−841.4492	−841.4648	0.000	1190.000	841.479	−841.435
16	0.0080	1110	−554.9897	−961.2942	−555.021	961.276	1110.000	0.000
17	0.0085	1050	−271.7498	−1014.2249	−909.337	524.982	742.440	742.485
18	0.0090	990	0.0000	−990.0000	−990.000	0.000	0.000	990.000
19	0.0095	930	240.7123	−898.3081	−805.393	−465.018	−657.633	657.586
20	0.0100	890	445.0095	−770.7571	−444.981	−770.773	−890.000	0.000
21	0.0105	850	601.0478	−601.0337	0.000	−850.000	−601.022	−601.060
22	0.0110	810	701.4868	−404.9895	405.022	−701.468	0.000	−810.000
23	0.0115	770	743.7659	−199.2798	666.851	−384.980	544.500	−544.444
24	0.0120	750	750.0000	0.0000	750.000	0.000	750.000	0.000
$\sum_{i=1}^{24}(\ )$		27,300	−4,979.3242	1,803.7673	343.270	−1,754.047	428.734	661.855
$\frac{1}{12}\sum_{i=1}^{24}$		2,275	−414.9436	150.3139	28.606	−146.171	35.728	55.155

1.117

i	t_i	x_i	$n=1$: $x_i \cos\frac{2\pi t_i}{0.6}$	$n=1$: $x_i \sin\frac{2\pi t_i}{0.6}$	$n=2$: $x_i \cos\frac{4\pi t_i}{0.6}$	$n=2$: $x_i \sin\frac{4\pi t_i}{0.6}$	$n=3$: $x_i \cos\frac{6\pi t_i}{0.6}$	$n=3$: $x_i \sin\frac{6\pi t_i}{0.6}$
1	0.025	9.00	8.69	2.33	7.79	4.50	6.36	6.36
2	0.050	17.00	14.72	8.50	8.50	14.72	0.00	17.00
3	0.075	23.00	16.26	16.26	0.00	23.00	−16.26	16.26
4	0.100	25.00	12.50	21.65	−12.50	21.65	−25.00	0.00
5	0.125	26.00	6.73	25.11	−22.52	13.00	−18.38	−18.38
6	0.150	28.00	0.00	28.00	−28.00	0.00	0.00	−28.00
7	0.175	33.00	−8.54	31.88	−28.58	−16.50	23.33	−23.33
8	0.200	35.00	−17.50	30.31	−17.50	−30.31	35.00	0.00
9	0.225	34.00	−24.04	24.04	0.00	−34.00	24.04	24.04
10	0.250	29.00	−25.11	14.50	14.50	−25.11	0.00	29.00
11	0. 75	24.00	−23.18	6.21	20.78	−12.00	−16.97	16.97
12	0.300	26.00	−26.00	0.00	26.00	0.00	−26.00	0.00
13	0.325	32.00	−30.91	−8.28	27.71	16.00	−22.63	−22.63
14	0.350	40.00	−34.64	−20.00	20.00	34.64	0.00	−40.00
15	0.375	18.00	−12.73	−12.73	0.00	18.00	12.73	−12.73
16	0.400	8.00	−4.00	−6.93	−4.00	6.93	8.00	0.00
17	0.425	−5.00	1.29	4.83	4.33	−2.50	−3.54	−3.54
18	0.450	−14.00	0.00	14.00	14.00	0.00	0.00	−14.00
19	0.475	−28.00	−7.25	27.05	24.25	14.00	19.80	−19.80
20	0.500	−37.00	−18.50	32.04	18.50	32.04	37.00	0.00
21	0.525	−33.00	−23.33	23.33	0.00	33.00	23.33	23.34
22	0.550	−29.00	−25.11	14.50	−14.50	25.11	0.00	29.00

23	0.575	-22.00	-21.25	5.69	-19.05	11.00	-15.56	15.56
24	0.600	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$\sum_{i=1}^{24}(\ )$		239.00	-241.90	282.30	39.72	147.18	45.26	-4.88
$\frac{1}{12}\sum_{i=1}^{24}(\ )$		19.92	-20.16	23.53	3.31	12.26	3.77	-0.41

1.118

```
%=========================================================================
%
%Program1.m
%Program for calling the subroutine FORIER
%
%=========================================================================
%Run "Program1.m" in MATLAB Command Window. Program1.m and forier.m should be
%in the same file folder, and set the path to this folder
%Following 6 lines contain problem-dependent data
n=16;
m=3;
time=0.32;
x=[9 13 17 29 43 59 63 57 49 35 35 41 47 41 13 7];
t=0.02:0.02:0.32;
%end of problem-dependent data
%Following line calls subroutine forier.m
[azero,a,b,xsin,xcos]=forier(n,m,time,x,t);
%following outputs data
fprintf('Fourier series expansion of the function x(t)\n\n');
fprintf('Data:\n\n');
fprintf('Number of data points in one cycle = %3.0f \n',n);
fprintf(' \n');
fprintf('Number of Fourier Coefficients required = %3.0f \n',m);
fprintf(' \n');
fprintf('Time period = %8.6e \n\n',time);
fprintf('Station i       ')
fprintf('Time at station i: t(i)         ')
fprintf('x(i) at t(i)')
for i=1:n
   fprintf('\n %8d%25.6e%27.6e ',i,t(i),x(i));
end
fprintf(' \n\n');
fprintf('Results of Fourier analysis:\n\n');
fprintf('azero=%8.6e \n\n',azero);
fprintf('values of i       a(i)                  b(i)\n');
for i=1:m
   fprintf('%10.0g   %8.6e%20.6e \n',i,a(i),b(i));
end
```

```
%=================================================================
%
%Subroutine forier.m
%
%=================================================================
function [azero,a,b,xsin,xcos]=forier(n,m,time,x,t)
pi=3.1416;
sumz=0.0;
for i=1:n
   sumz=sumz+x(i);
end
azero=2.0*sumz/n;
for ii=1:m
   sums=0.0;
   sumc=0.0;
   for i=1:n
      theta=2.0*pi*t(i)*ii/time;
      xcos(i)=x(i)*cos(theta);
      xsin(i)=x(i)*sin(theta);
      sums=sums+xsin(i);
      sumc=sumc+xcos(i);
   end
   a(ii)=2.0*sumc/n;
   b(ii)=2.0*sums/n;
end
```

```
>> program1
Fourier series expansion of the function x(t)

Data:

Number of data points in one cycle =  16

Number of Fourier Coefficients required =   3

Time period = 3.200000e-001

Station i       Time at station i: t(i)        x(i) at t(i)
       1            2.000000e-002              9.000000e+000
       2            4.000000e-002              1.300000e+001
       3            6.000000e-002              1.700000e+001
       4            8.000000e-002              2.900000e+001
       5            1.000000e-001              4.300000e+001
       6            1.200000e-001              5.900000e+001
       7            1.400000e-001              6.300000e+001
       8            1.600000e-001              5.700000e+001
       9            1.800000e-001              4.900000e+001
      10            2.000000e-001              3.500000e+001
      11            2.200000e-001              3.500000e+001
      12            2.400000e-001              4.100000e+001
      13            2.600000e-001              4.700000e+001
      14            2.800000e-001              4.100000e+001
      15            3.000000e-001              1.300000e+001
      16            3.200000e-001              7.000000e+000

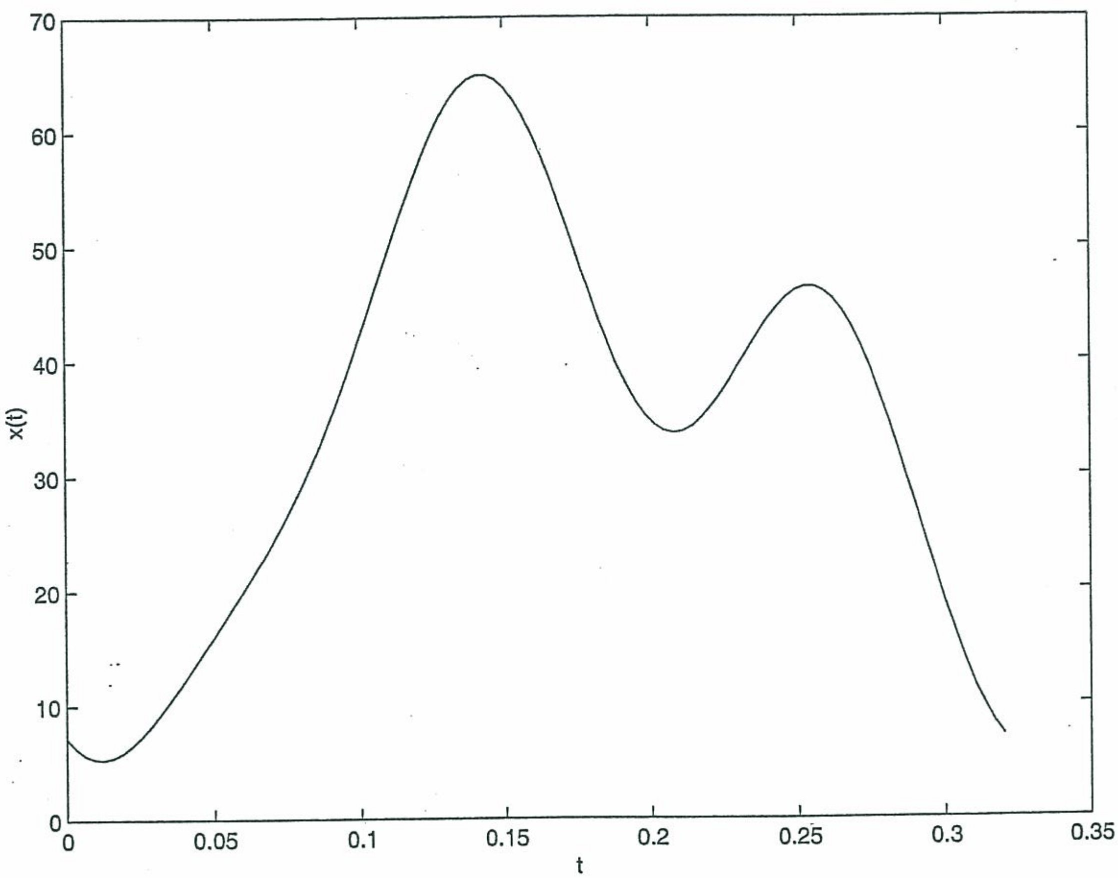
Results of Fourier analysis:

azero=6.975000e+001

values of i        a(i)                b(i)
        1    -2.084870e+001      -3.915985e+000
        2    -1.456887e+000      -1.144979e+001
        3    -5.402900e+000       3.353473e+000
```

1.119

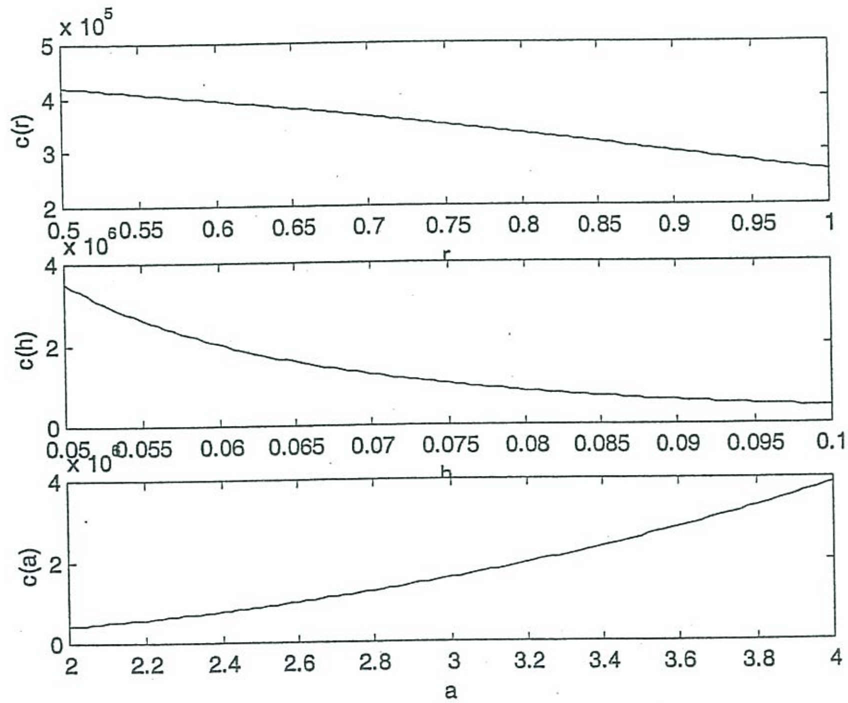
```
% Ex1_119.m
for i = 1: 101
    t(i) = 0.32*(i-1)/100;
    x(i) = 34.875 - 20.8487*cos(19.635*t(i)) - 3.9160*sin(19.635*t(i))...
        - 1.4569*cos(39.27*t(i)) - 11.4498*sin(39.27*t(i))...
        - 5.4029*cos(58.905*t(i)) + 3.3535*sin(58.905*t(i));
end
plot(t,x)
xlabel('t');
ylabel('x(t)');
```



1.120

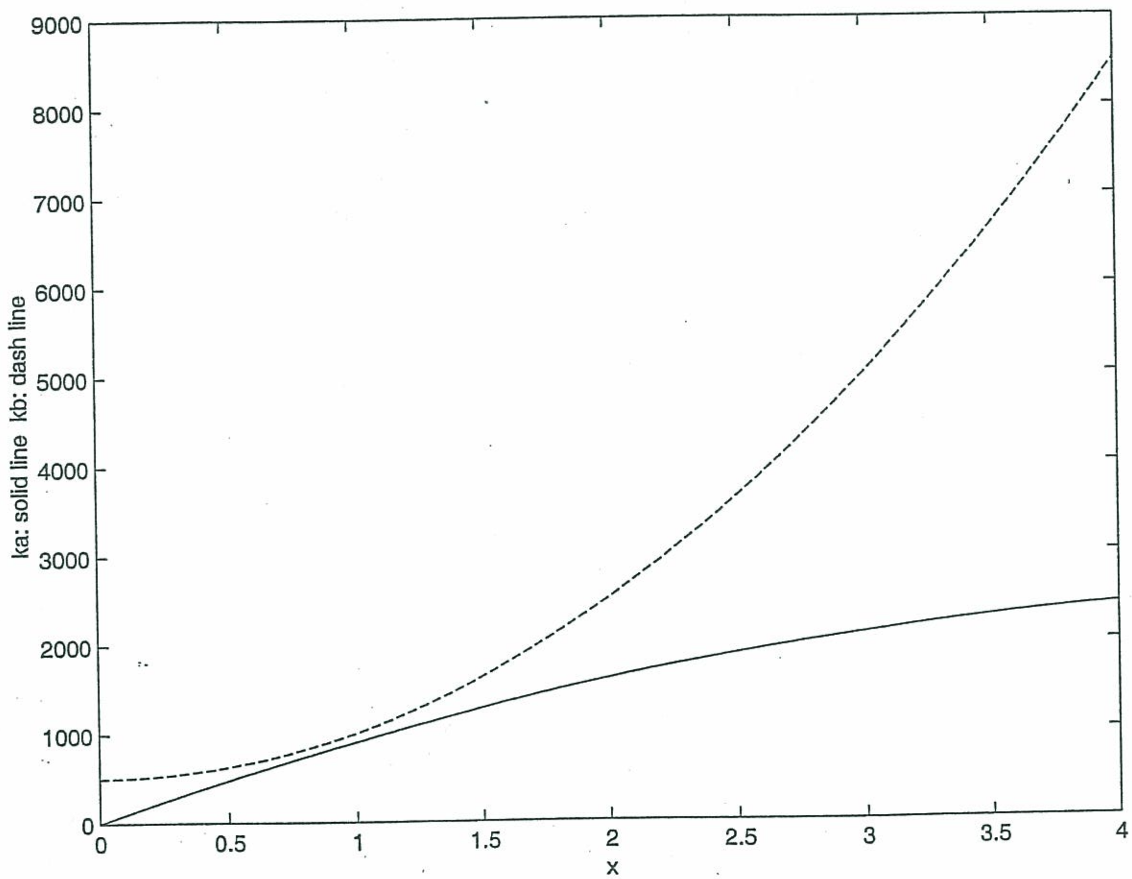
```
% Ex1_1.120.m
u = 0.3445;
l = 10;
h0 = 0.1;
a0 = 2;
r0 = 0.5;
% First case, r changes
for i = 1:101
    r(i) = 0.5 + (i-1)*0.5/100;
    c1(i) = ( 6*pi*u*l/(h0^3) ) * ( (a0 - h0/2)^2 - r(i)^2 )...
        * ( (a0^2-r(i)^2)/(a0-h0/2) - h0 );
end
```

```
% Second case, h changes
for i = 1:101
    h(i) = 0.05 + (i-1)*0.05/100;
    c2(i) = ( 6*pi*u*l/(h(i)^3) ) * ( (a0 - h(i)/2)^2 - r0^2 )...
        * ( (a0^2-r0^2)/(a0-h(i)/2) - h(i) );
end
% Third case, a changes
for i = 1:101
    a(i) = 2 + (i-1)*2/100;
    c3(i) = ( 6*pi*u*l/(h0^3) ) * ( (a(i) - h0/2)^2 - r0^2 )...
        * ( (a(i)^2-r0^2)/(a(i)-h0/2) - h0 );
end
subplot(311);
plot(r,c1);
xlabel('r');
ylabel('c(r)');
subplot(312);
plot(h,c2);
xlabel('h');
ylabel('c(h)');
subplot(313);
plot(a,c3);
xlabel('a');
ylabel('c(a)');
```



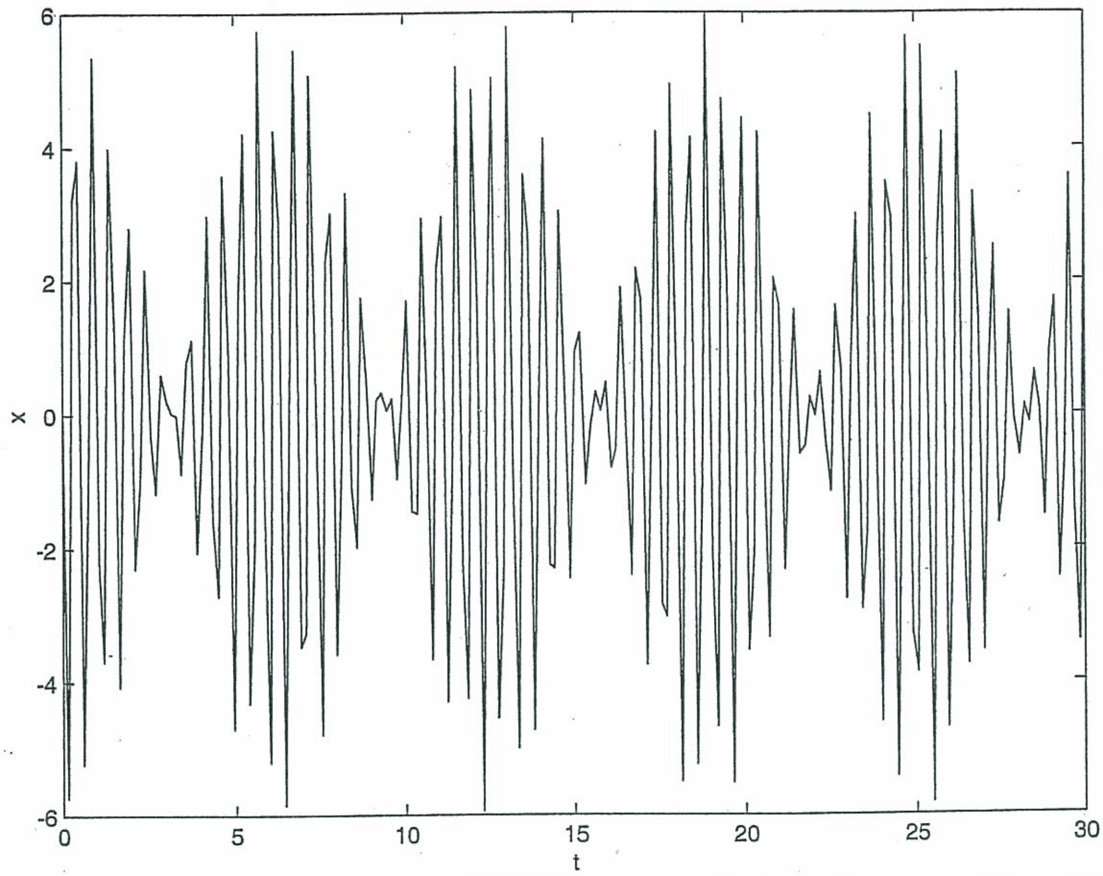
1.121

```
% Ex1_121.m
for i = 1:101
    x(i) = (i-1)*4/100;
    ka(i) = 1000*x(i) - 100*x(i)^2;
    kb(i) = 500 + 500 *x(i)^2;
end
plot(x,ka);
hold on
plot(x,kb, '--');
xlabel('x');
ylabel('ka: solid line  kb: dash line');
```



1.122

```
% Ex1_122.m
for i = 1:201
    t(i) = (i-1)*30/200;
    x1(i) = 3*sin(30*t(i));
    x2(i) = 3*sin(29*t(i));
    x(i) = x1(i) + x2(i);
end
plot(t,x);
xlabel('t');
ylabel('x');
```


6
4
2
x 0
-2
-4
-6
0
5
10
15
20
25
30
t

1.123

$$x_p = r + l - r\cos\theta - l\cos\phi = r + l - r\cos\omega t - l\sqrt{1-\sin^2\phi} \qquad (E_1)$$

But $l\sin\phi = r\sin\theta$, $\cos\phi = \left(1 - \frac{r^2}{l^2}\sin^2\omega t\right)^{\frac{1}{2}}$ $\qquad (E_2)$

Using (E_2) in (E_1), $x_p = r + l - r\cos\omega t - l\left(1 - \frac{r^2}{l^2}\sin^2\omega t\right)^{\frac{1}{2}}$ $\qquad (E_3)$

Let $\frac{r}{l}$ = small $(< \frac{1}{4})$. Using $\sqrt{1-\epsilon} \approx 1 - \frac{1}{2}\epsilon$, (E_3) becomes

$$x_p \approx r\left(1 + \frac{r}{2l}\right) - r\left(\cos\omega t + \frac{r}{4l}\cos 2\omega t\right) \qquad (E_4)$$

(a) Eq. (E_4) gives $\quad y_p = x_p - r\left(1 + \frac{r}{2l}\right) \simeq -r\left(\cos\omega t + \frac{1}{4}\frac{r}{l}\cos 2\omega t\right)$ --- (E_5)

If $\frac{r}{l}$ is very small, $y_p \simeq -r\cos\omega t \Rightarrow$ harmonic motion.

(b) To have amplitude of second harmonic smaller than that of first harmonic in Eq. (E_5), we need to have

$$\frac{1}{4}\frac{r}{l} \le \frac{1}{25}, \quad \text{i.e.,} \quad \frac{r}{l} \le \frac{4}{25}, \quad \text{i.e.,} \quad \frac{l}{r} \ge 6.25$$

Once the amplitude of second harmonic is smaller by a factor of 25, the amplitudes of higher harmonics arising from the expansion of square-root-term in (E_3) are expected to be still smaller.

1.124 Unbalanced force developed $= P = 2\,m\,\omega^2\,r\cos\omega t$, range of force = 0 - 100 N, range of frequency = 25 - 50 Hz = 157.08 - 314.16 rad/sec.
Parameters to be determined: m, r, ω.
Let r = 0.1 m. To generate 100 N force at 25 Hz, set:

$$P_{max} = 100 = 2\,m\,(157.08)^2\,(0.1)$$

which gives

$$m = \frac{100}{2\,(157.08)^2\,(0.1)} = 0.0202641 \text{ kg} = 20.2641 \text{ g}$$

To generate 100 N force at 50 Hz, set:

$$P_{max} = 100 = 2\,m\,(314.16)^2\,(0.1)$$

which yields

$$m = \frac{100}{2\,(314.16)^2\,(0.1)} = 0.0050660 \text{ kg} = 5.0660 \text{ g}$$

1.125 Goal: Weight to be maintained at 10 ± 0.1 lb/min
Parameters to be determined: Angular velocity of crank (ω), lengths of crank and connecting rod, dimensions of the wedge, dimensions of the orifice in the hopper, dimensions of the actuating rod, and dimensions of the lever arrangement.
Given: Density of the material in the hopper.
Procedure:
Select ω based on available motor. Determine the dimensions of the orifice in the hopper which delivers approximately 10 lb/min (assuming continuous flow of material). For trial dimensions of the wedge, determine the increase/decrease in the size (diameter) of the orifice. Choose the final dimensions of the wedge such that the material flow rate delivered by the orifice lies within the specified range.

1.126 Force to be applied = 200 lb, frequency = 50 Hz = 314.16 rad/sec.

Procedure:

1. Select a motor that provides, either directly or through a gear system, the desired frequency. Assume that it is connected to the cam.
2. D ~~d~~etermine the sizes and dimensions of the plate cam and the roller.
3. Choose the dimensions of the follower.
4. Select the weight as 200 lb. From the geometry, determine the range of displacement (vertical motion) of the weight.
5. Determine the force exerted due to the falling weight.

1.127 Considerations to be taken in the design of vibratory bowl feeders:

1. Suitable design of the electromagnet and its coil.
2. Radius of the bowl and the pitch of the spiral (helical) delivery track.
3. Tooling to be fixed along the spiral track to reject the defective or out-of-tolerance or incorrectly oriented parts.
4. Design of elastic supports.
5. Size and location of the outlet.

1.128 Axial spring constant of each tube $= k = \dfrac{A\,E}{\ell}$.

Let diameter of each tube be 0.01 m (1 cm) with thickness 0.001 m (1 mm). Then

$$A = \frac{\pi}{4}\left(D^2 - d^2\right) = \frac{\pi}{4}\left(0.01^2 - 0.008^2\right) = 28.27\ (10^{-6})\ m^2$$

This gives

$$k = \frac{(28.27\ (10^{-6}))\ (2.07\ (10^{11}))}{2} = 29.26\ (10^5)\ N/m$$

Since 76 tubes are in parallel, we have the total axial stiffness as:

$$k_{eq} = 76\ k = (76)\ (29.26\ (10^5)) = 222.38\ (10^6)\ N/m$$

The polar area moment of inertia of each tube is

$$J = \frac{\pi}{32}\left(D^4 - d^4\right) = \frac{\pi}{32}\left(0.01^4 - 0.008^4\right) = 580\ (10^{-8})\ m^4$$

Torsional stiffness of each tube is given by

$$\frac{G\,J}{\ell} = \frac{(79.6154\ (10^9))\ (580\ (10^{-8}))}{2} = 231\ (10^3)\ N\text{—}m/rad$$

For 76 tubes in parallel, equivalent torsional stiffness will be:

$$k_{t_{eq}} = (76)\ (231\ (10^3)) = 17.56\ (10^6)\ N\text{—}m/rad$$

Chapter 2

Free Vibration of Single Degree of Freedom Systems

2.1

$\delta_{st} = 5 \times 10^{-3}$ m

$$\omega_n = \left(\frac{g}{\delta_{st}}\right)^{1/2} = \left(\frac{9.81}{5 \times 10^{-3}}\right)^{1/2} = 44.2945 \text{ rad/sec} = 7.0497 \text{ Hz}$$

2.2

$$\tau_n = 0.21 \text{ sec} = 2\pi\sqrt{\frac{m}{k}}, \quad \sqrt{m} = 0.21\sqrt{k}/2\pi$$

(i) $$(\tau_n)_{new} = \frac{2\pi\sqrt{m}}{\sqrt{k_{new}}} = \frac{2\pi\sqrt{m}}{\sqrt{1.5k}} = \frac{2\pi\left(\frac{0.21\sqrt{k}}{2\pi}\right)}{\sqrt{1.5k}} = 0.1715 \text{ sec.}$$

(ii) $$(\tau_n)_{new} = \frac{2\pi\sqrt{m}}{\sqrt{k_{new}}} = \frac{2\pi\sqrt{m}}{\sqrt{0.5k}} = 2\pi\left(\frac{0.21\sqrt{k}}{2\pi}\right)\frac{1}{\sqrt{0.5k}} = 0.2970 \text{ sec.}$$

2.3

$$\omega_n = 62.832 \text{ rad/sec} = \sqrt{\frac{k}{m}}, \quad \sqrt{m} = \sqrt{k}/62.832$$

When spring constant is reduced, ω_n decreases.

$$(\omega_n)_{new} = 0.55\,\omega_n = 34.5576 \text{ rad/sec} = \sqrt{\frac{k_{new}}{m_{new}}} = \sqrt{\frac{k-800}{m}}$$

$$\sqrt{\frac{k-800}{k}} \times 62.836 = 34.5576, \quad \sqrt{\frac{k-800}{k}} = 0.55$$

$$\frac{k-800}{k} = (0.55)^2 = 0.3025$$

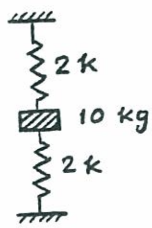
$$k = 1146.9534 \text{ N/m}$$

$$\sqrt{m} = \sqrt{k}/62.832; \qquad m = k/62.832^2 = \frac{1146.9534}{3947.8602}$$

$$m = 0.2905 \text{ kg}$$

2.4

$$k = 100/\left(\frac{10}{1000}\right) = 10000 \text{ N/m}$$



$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{4k}{m}} = \left(\frac{4 \times 10^4}{10}\right)^{1/2}$$

$$= 63.2456 \text{ rad/sec}$$

$$\tau_n = \frac{2\pi}{\omega_n} = \frac{6.2832}{63.2456} = 0.0993 \text{ sec}$$

2.5 $m = \frac{2000}{386.4}$.

Let $\omega_n = 7.5$ rad/sec.

$$\omega_n = \sqrt{\frac{k_{eq}}{m}}$$

$$k_{eq} = m\,\omega_n^2 = \left(\frac{2000}{386.4}\right)(7.5)^2 = 291.1491 \text{ lb/in} = 4\text{ k}$$

where k is the stiffness of the air spring.

Thus $k = \frac{291.1491}{4} = 72.7873$ lb/in.

2.6 $x = A\cos(\omega_n t - \phi_0)$, $\dot{x} = -\omega_n A \sin(\omega_n t - \phi_0)$,

$\ddot{x} = -\omega_n^2 A \cos(\omega_n t - \phi_0)$

(a) $\omega_n A = 0.1$ m/sec ; $\tau_n = \frac{2\pi}{\omega_n} = 2$ sec, $\omega_n = 3.1416$ rad/sec

$A = 0.1/\omega_n = 0.03183$ m

(d) $x_0 = x(t=0) = A\cos(-\phi_0) = 0.02$ m

$\cos(-\phi_0) = \frac{0.02}{A} = 0.6283$

$\phi_0 = 51.0724^\circ$

(b) $\dot{x}_0 = \dot{x}(t=0) = -\omega_n A \sin(-\phi_0) = -0.1\sin(-51.0724^\circ)$

$= 0.07779$ m/sec

(c) $\ddot{x}\big|_{max} = \omega_n^2 A = (3.1416)^2(0.03183) = 0.314151$ m/sec^2

2.7 For small angular rotation of bar PQ about P,

$$\frac{1}{2}(k_{12})_{eq}(\theta l_3)^2 = \frac{1}{2}k_1(\theta l_1)^2 + \frac{1}{2}k_2(\theta l_2)^2$$

i.e., $(k_{12})_{eq} = (k_1 l_1^2 + k_2 l_2^2)/l_3^2$

Let k_{eq} = overall spring constant at Q.

$$\frac{1}{k_{eq}} = \frac{1}{(k_{12})_{eq}} + \frac{1}{k_3}$$

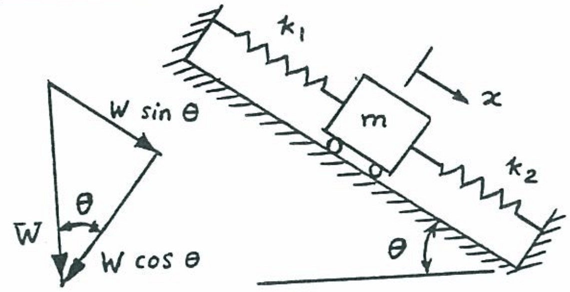
$$k_{eq} = \frac{(k_{12})_{eq}\,k_3}{(k_{12})_{eq} + k_3} = \frac{\left\{k_1\left(\frac{l_1}{l_3}\right)^2 + k_2\left(\frac{l_2}{l_3}\right)^2\right\}k_3}{k_1\left(\frac{l_1}{l_3}\right)^2 + k_2\left(\frac{l_2}{l_3}\right)^2 + k_3}$$

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k_1 k_2 l_1^2 + k_2 k_3 l_2^2}{m\,(k_1 l_1^2 + k_2 l_2^2 + k_3 l_3^2)}}$$

2.8 $m = 2000$ kg , $\delta_{st} = 0.02$ m

$$\omega_n = (g/\delta_{st})^{1/2} = \left(\frac{9.81}{0.02}\right)^{1/2} = 22.1472 \text{ rad/sec}$$

2.9 Let x be measured from the position of mass at which the springs are unstretched.



Equation of motion is

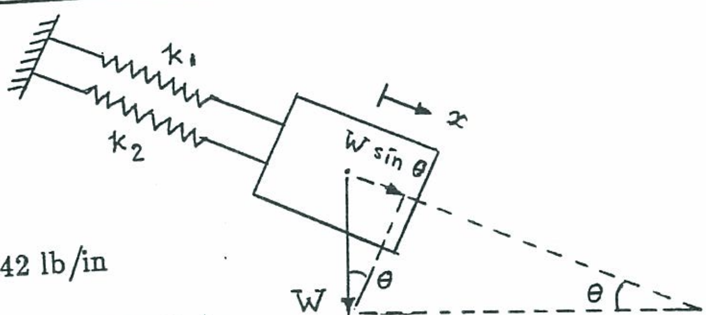
$$m\ddot{x} = -k_1(x+\delta_{st}) - k_2(x+\delta_{st}) + W\sin\theta \quad ---- \quad (E_1)$$

where $\delta_{st}(k_1+k_2) = W\sin\theta$.

Thus Eq. (E_1) becomes $m\ddot{x} + (k_1+k_2)\,x = 0 \Rightarrow \omega_n = \sqrt{\frac{k_1+k_2}{m}}$.

2.10

$$k_1 = \frac{A_1 E_1}{\ell_1} = \frac{\frac{\pi}{4}(0.05)^2\,(30\,(10^6))}{30\,(12)} = 163.6250 \text{ lb/in}$$



$$k_2 = \frac{A_2 E_2}{\ell_2} = \frac{163.625\,(25)}{30} = 136.3542 \text{ lb/in}$$

$k_{eq} = k_1 + k_2 = 163.6250 + 136.3542 = 299.9792$ lb/in

Let x be measured from the unstretched length of the springs. The equation of motion is:

$$m\ddot{x} = -(k_1+k_2)(x+\delta_{st}) + W\sin\theta$$

where $(k_1+k_2)\,\delta_{st} = W\sin\theta$

$$\text{i.e.,}\quad m\ddot{x} + (k_1+k_2)\,x = 0$$

Thus the natural frequency of vibration of the cart is given by

$$\omega_n = \sqrt{\frac{k_1+k_2}{m}} = \sqrt{\frac{299.9792\,(386.4)}{5000}} = 4.8148 \text{ rad/sec}$$

2.11 Weight of electronic chassis = 500 N. To be able to use the unit in a vibratory environment with a frequency range of 0 - 5 Hz, its natural frequency must be away from the frequency of the environment. Let the natural frequency be $\omega_n = 10$ Hz = 62.832 rad/sec. Since

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = 62.832$$

we have

$$k_{eq} = m\,\omega_n^2 = \left(\frac{500}{9.81}\right)(62.832)^2 = 20.1857\,(10^4)\text{ N/m} \equiv 4\text{ k}$$

so that k = spring constant of each spring = 50,464.25 N/m. For a helical spring,

$$k = \frac{G\,d^4}{8\,n\,D^3}$$

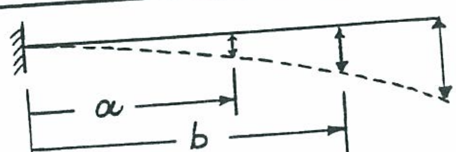
Assuming the material of springs as steel with $G = 80\,(10^9)$ Pa, $n = 5$ and $d = 0.005$ m, we find

$$k = 50{,}464.25 = \frac{80\,(10^9)\,(0.005)^4}{8\,(5)\,D^3}$$

This gives

$$D^3 = \frac{1250\,(10^{-3})}{50464.25} = 24{,}770.0\,(10^{-9}) \quad\text{or}\quad D = 0.0291492\text{ m} = 2.91492\text{ cm}$$

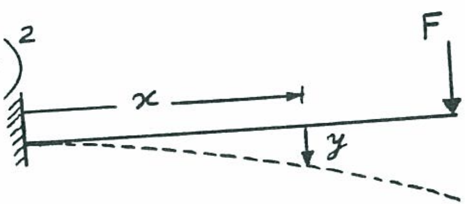
2.12 (i) with springs k_1 and k_2:



Let y_a, y_b, y_ℓ be deflections of beam at distances a, b, ℓ from fixed end.

$$\frac{1}{2}(k_{12})_{eq}\; y_\ell^2 = \frac{1}{2} k_1\, y_a^2 + \frac{1}{2} k_2\, y_b^2$$

i.e.,
$$(k_{12})_{eq} = k_1\left(\frac{y_a}{y_\ell}\right)^2 + k_2\left(\frac{y_b}{y_\ell}\right)^2$$



$$y = \frac{F x^2}{6EI}(3\ell - x)$$

$$@\; x = a, \quad y_a = \frac{F a^2}{6EI}(3\ell - a)$$

$$@\; x = b, \quad y_b = \frac{F b^2}{6EI}(3\ell - b)$$

$$@\; x = \ell, \quad y_\ell = \frac{F \ell^3}{3EI}$$

$$\omega_n = \left[\frac{k_1\, k_3 \left(\frac{y_a}{y_\ell}\right)^2 + k_2\, k_3 \left(\frac{y_b}{y_\ell}\right)^2}{m\left\{k_1\left(\frac{y_a}{y_\ell}\right)^2 + k_2\left(\frac{y_b}{y_\ell}\right)^2 + k_{beam}\right\}}\right]^{\frac{1}{2}} \quad \text{where } k_{beam} = \frac{3EI}{\ell^3}$$

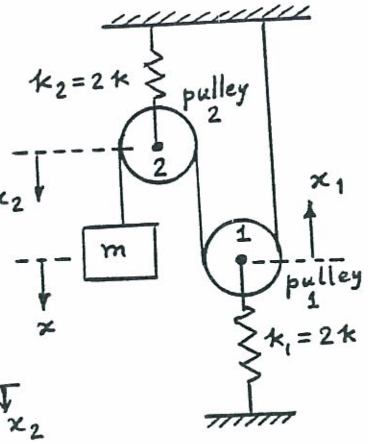
$$= \left[\frac{k_1(3EI)\,a^4(3\ell-a)^2 + k_2(3EI)\,b^4(3\ell-b)^2}{m\ell^3\left\{k_1 a^4(3\ell-a)^2 + k_2 b^4(3\ell-b)^2 + 12\,EI\,\ell^3\right\}}\right]^{\frac{1}{2}}$$

(ii) without springs k_1 and k_2:

$$\omega_n = \sqrt{\frac{k_{beam}}{m}} = \sqrt{\frac{3EI}{m\ell^3}}$$

2.13 Let x_1, x_2 = displacements of pulleys 1, 2

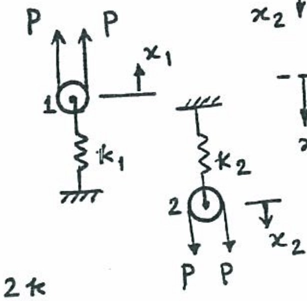
$$x = 2x_1 + 2x_2 \quad ---- (E_1)$$



Let P = tension in rope.

For equilibrium of pulley 1,

$$2P = k_1 x_1 \quad ---- (E_2)$$



For equilibrium of pulley 2,

$$2P = k_2 x_2 \quad ---- (E_3)$$

where $\frac{1}{k_1} = \frac{1}{4k} + \frac{1}{4k} = \frac{1}{2k}$; $k_1 = 2k$

and $k_2 = k + k = 2k$

Combining Eqs. (E_1) to (E_3):

$$x = 2x_1 + 2x_2 = 2\left(\frac{2P}{k_1}\right) + 2\left(\frac{2P}{k_2}\right) = 4P\left(\frac{1}{2k} + \frac{1}{2k}\right) = \frac{4P}{k}$$

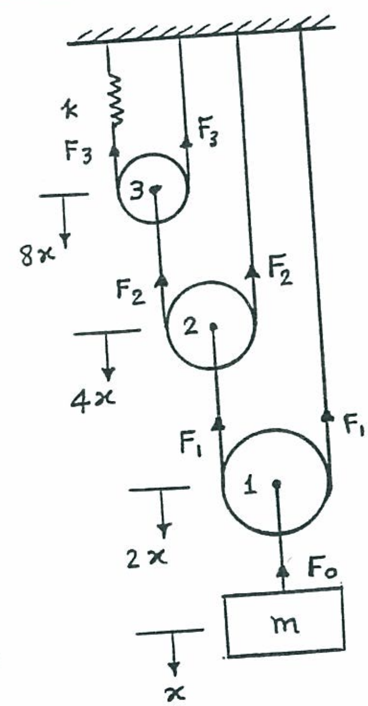
Let k_{eq} = equivalent spring constant of the system:

$$k_{eq} = \frac{P}{x} = \frac{k}{4}$$

Equation of motion of mass m: $\quad m\ddot{x} + k_{eq}\, x = 0$

$$\therefore \omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k}{4m}}$$

2.14 For a displacement of x of mass m, pulleys 1, 2 and 3 undergo displacements of 2x, 4x and 8x, respectively. The equation of motion of mass m can be written as



$$m\ddot{x} + F_0 = 0 \qquad (1)$$

where $F_0 = 2F_1 = 4F_2 = 8F_3$ as shown in figure.

Since $F_3 = (8x)\,k$, Eq. (1) can be rewritten as

$$m\ddot{x} + 8F_3 = 8(8k) = 0 \qquad (2)$$

from which we can find

$$\omega_n = \sqrt{\frac{64k}{m}} = 8\sqrt{\frac{k}{m}} \qquad (3)$$

2.15 (a) $\omega_n = \sqrt{4k/M}$

(b) $\omega_n = \sqrt{4k/(M+m)}$

Initial conditions:

velocity of falling mass $m = v = \sqrt{2gl}$ $\left(\because\ v^2 - \overset{0}{u^2} = 2gl\right)$

$x = 0$ at static equilibrium position.

$$x_0 = x(t=0) = -\frac{\text{weight}}{k_{eq}} = -\frac{mg}{4k}$$

Conservation of momentum: $(M+m)\dot{x}_0 = m v = m\sqrt{2gl}$

$$\dot{x}_0 = \dot{x}(t=0) = \frac{m}{M+m}\sqrt{2gl}$$

Complete solution: $x(t) = A_0 \sin(\omega_n t + \phi_0)$

where $A_0 = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2} = \sqrt{\frac{m^2 g^2}{16k^2} + \frac{m^2 gl}{2k(M+m)}}$

and $\phi_0 = \tan^{-1}\left(\frac{x_0\omega_n}{\dot{x}_0}\right) = \tan^{-1}\left(\frac{-\sqrt{g}}{\sqrt{8lk(M+m)}}\right)$

2.16 (a) Velocity of anvil = v = 50 ft/sec = 600 in/sec. x = 0 at static equilibrium position.

$$x_0 = x(t{=}0) = -\frac{\text{weight}}{k_{eq}} = -\frac{m\,g}{4\,k}$$

Conservation of momentum:

$$(M+m)\,\dot{x}_0 = m\,v \quad \text{or} \quad \dot{x}_0 = \dot{x}(t{=}0) = \frac{m\,v}{M+m}$$

Natural frequency:

$$\omega_n = \sqrt{\frac{4\,k}{M+m}}$$

Complete solution:

$$x(t) = A_0 \sin(\omega_n\,t + \phi_0)$$

where

$$A_0 = \left\{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2\right\}^{\frac{1}{2}} = \left\{\frac{m^2\,g^2}{16\,k^2} + \frac{m^2\,v^2}{(M+m)\,4\,k}\right\}^{\frac{1}{2}}$$

and

$$\phi_0 = \tan^{-1}\left(\frac{x_0\,\omega_n}{\dot{x}_0}\right) = \tan^{-1}\left(-\frac{m\,g}{4\,k}\sqrt{\frac{4\,k}{(M+m)}}\,\frac{(M+m)}{m\,v}\right) = \tan^{-1}\left(-\frac{g\sqrt{M+m}}{v\sqrt{4\,k}}\right)$$

Since v = 600, m = 12/386.4, M = 100/386.4, k = 100, we find

$$A_0 = \left\{ \left(\frac{12\ (386.4)}{4\ (100)\ (386.4)} \right)^2 + \left(\frac{12\ (600)}{386.4} \right)^2 \frac{386.4}{112\ (400)} \right\}^{\frac{1}{2}} = 1.7308 \text{ in}$$

$$\phi_0 = \tan^{-1} \left(- \frac{386.4 \sqrt{112}}{\sqrt{386.4}\ (600) \sqrt{400}} \right) = \tan^{-1}(-0.01734) = -0.9934 \text{ deg}$$

(b) x = 0 at static equilibrium position: $x_0 = x(t{=}0) = 0$. Conservation of momentum gives:

$$M \dot{x}_0 = m\,v \quad \text{or} \quad \dot{x}_0 = \dot{x}(t{=}0) = \frac{m\,v}{M}$$

Complete solution:

$$x(t) = A_0 \sin(\omega_n t + \phi_0)$$

where

$$A_0 = \left\{ x_0^2 + \left(\frac{\dot{x}_0}{\omega_n} \right)^2 \right\}^{\frac{1}{2}} = \left\{ \frac{m^2\ v^2\ (M)}{M^2\ 4\ k} \right\}^{\frac{1}{2}} = \frac{m\ v}{\sqrt{4\,k\,M}} = \frac{12\ (600) \sqrt{386.4}}{386.4 \sqrt{4\ (100)\ (100)}} = 1.8314 \text{ in}$$

$$\phi_0 = \tan^{-1} \left(\frac{x_0\ \omega_n}{\dot{x}_0} \right) = \tan^{-1}(0) = 0$$

2.17 $k_1 = \dfrac{3 E_1 I_1}{l_1^3}$ (at tip) ; $k_2 = \dfrac{48 E_2 I_2}{l_2^3}$ (at middle)

$k_{eq} = k_1 + k_2$

$\omega_n = \sqrt{\dfrac{k_{eq}}{m}} = \sqrt{\left(\dfrac{3 E_1 I_1}{l_1^3} + \dfrac{48 E_2 I_2}{l_2^3} \right) \dfrac{g}{W}}$

2.18 $k = \dfrac{AE}{l} = \dfrac{\left\{ \frac{\pi}{4} (0.01)^2 \right\} \left\{ 2.07 \times 10^{11} \right\}}{20} = 0.8129 \times 10^6$ N/m

m = 1000 kg

$\omega_n = \sqrt{\dfrac{k}{m}} = \left(\dfrac{0.8129 \times 10^6}{1000} \right)^{1/2} = 28.5114$ rad/sec

$\dot{x}_0 = 2$ m/s, $x_0 = 0$ (suddenly stopped while it has velocity)

Period of ensuing vibration $= \tau_n = \dfrac{2\pi}{\omega_n} = \dfrac{2\pi}{28.5114} = 0.2204$ sec

Amplitude $= A = \dot{x}_0 / \omega_n = 2/28.5114 = 0.07015$ m

---

2.19 $\omega_n = 2$ Hz $= 12.5664$ rad/sec $= \sqrt{\dfrac{k}{m}}$

$\sqrt{k} = 12.5664 \sqrt{m}$

$\omega_n' = \sqrt{\dfrac{k'}{m'}} = \sqrt{\dfrac{k}{m+1}} = 6.2832$ rad/sec

$$\sqrt{k} = 6.2832\sqrt{m+1}$$
$$= 12.5664\sqrt{m}$$

$$\sqrt{m+1} = 2\sqrt{m}\ , \quad m = \frac{1}{3}\ \text{kg}$$

$$k = (12.5664)^2\ m = 52.6381\ \text{N/m}$$

(2.20) Cable stiffness $= k = \dfrac{A\,E}{\ell} = \dfrac{1}{4}\left(\dfrac{\pi}{4}(0.01)^2\right) 2.07\,(10^{11}) = 4.0644\,(10^6)$ N/m

$$\tau_n = 0.1 = \frac{1}{f_n} = \frac{2\pi}{\omega_n}$$

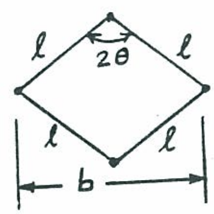
$$\omega_n = \frac{2\pi}{0.1} = 20\pi = \sqrt{\frac{k}{m}}$$

Hence

$$m = \frac{k}{\omega_n^2} = \frac{4.0644\,(10^6)}{(20\pi)^2} = 1029.53\ \text{kg}$$

(2.21) $b = 2\ell\sin\theta$

Neglect masses of links.



(a) $k_{eq} = k\left(\dfrac{4\ell^2 - b^2}{b^2}\right) = k\left(\dfrac{4\ell^2 - 4\ell^2\sin^2\theta}{4\ell^2\sin^2\theta}\right)$

$$= k\left(\frac{\cos^2\theta}{\sin^2\theta}\right)$$

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k\,g\ \text{cosec}^2\theta}{W}}$$ (from solution of problem 1.8)

(b) $\omega_n = \sqrt{\dfrac{k\,g}{W}}$ since $k_{eq} = k$.

(2.22) $y = \sqrt{\ell^2 - (\ell\sin\theta - x)^2} - \ell\cos\theta = \sqrt{\ell^2\cos^2\theta - x^2 + 2\,\ell\,x\sin\theta} - \ell\cos\theta$

$$= \ell\cos\theta\sqrt{1 - \frac{x^2}{\ell^2\cos^2\theta} + \frac{2\,\ell\,x\sin\theta}{\ell^2\cos^2\theta}} - \ell\cos\theta$$

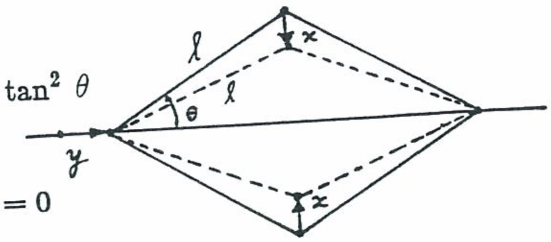
$$\frac{1}{2}k_{eq}\,x^2 = \frac{1}{2}k_1\,y^2 + \frac{1}{2}k_2\,y^2$$

where

$$y \approx \ell\cos\theta\left(1 - \frac{1}{2}\,\frac{x^2}{\ell^2\cos^2\theta} + \frac{1}{2}\,\frac{2\,\ell\,x\sin\theta}{\ell^2\cos^2\theta}\right) - \ell\cos\theta$$
$$\approx \frac{x\sin\theta}{\cos\theta} = x\tan\theta$$

Thus k_{eq} can be expressed as

$$k_{eq} = (k_1 + k_2)\tan^2\theta$$



Equation of motion:

$$m\ddot{x} + k_{eq}\,x = 0$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{(k_1+k_2)\,g}{W}}\tan\theta$$

(2.23) (a) Neglect masses of rigid links. Let x = displacement of W. Springs are in series.

$$k_{eq} = \frac{k}{2}$$

Equation of motion:

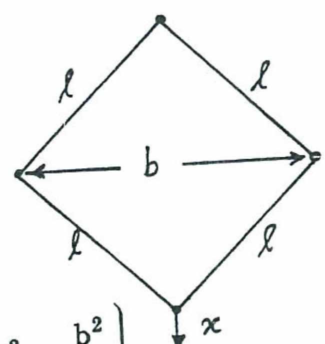
$$m\ddot{x} + k_{eq}\,x = 0$$

Natual frequency:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k}{2\,m}}$$

(b) Under a displacement of x of mass, each spring will be compressed by an an amount:

$$x_s = x\,\frac{2}{b}\sqrt{\ell^2 - \frac{b^2}{4}}$$



Equivalent spring constant:

$$\frac{1}{2}\,k_{eq}\,x^2 = 2\left(\frac{1}{2}\,k\,x_s^2\right)$$

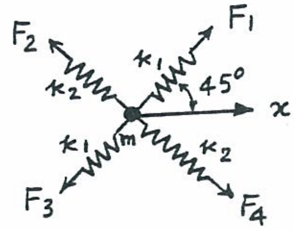
$$\text{or}\quad k_{eq} = 2\,k\left(\frac{x_s}{x}\right)^2 = 2\,k\left(\frac{4}{b^2}\right)\left(\ell^2 - \frac{b^2}{4}\right) = \frac{8\,k}{b^2}\left(\ell^2 - \frac{b^2}{4}\right)$$

Equation of motion:

$$m\ddot{x} + k_{eq}\,x = 0$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{8\,k}{b^2\,m}\left(\ell^2 - \frac{b^2}{4}\right)}$$



2.24 $F_1 = F_3 = k_1\, x \cos 45°$

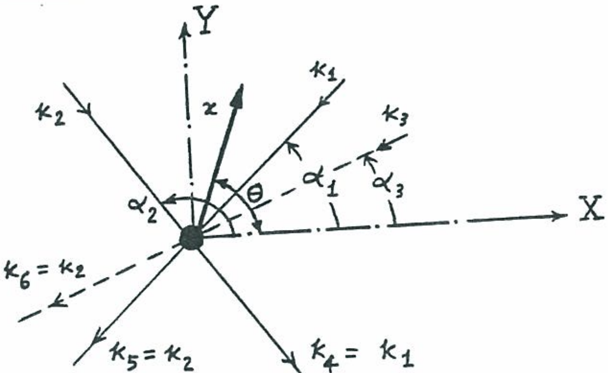
$F_2 = F_4 = k_2\, x \cos 135°$

$$F = \text{force along } x = F_1 \cos 45° + F_2 \cos 135° + F_3 \cos 45° + F_4 \cos 135°$$

$$= 2x\,(k_1 \cos^2 45° + k_2 \cos^2 135°)$$

$$k_{eq} = \frac{F}{x} = 2\left(\frac{k_1}{2} + \frac{k_2}{2}\right) = k_1 + k_2$$

Equation of motion: $m\ddot{x} + (k_1 + k_2)\,x = 0$



2.25 Let α_i denote the angle made by i^{th} spring with respect to X axis.

Let x = displacement of mass along the direction defined by θ.

If k_{eq} = equivalent spring constant, the equivalence of potential energies gives

$$\frac{1}{2} k_{eq}\, x^2 = \frac{1}{2} \sum_{i=1}^{6} k_i \left\{x \cos(\theta - \alpha_i)\right\}^2$$

$$k_{eq} = \sum_{i=1}^{6} k_i \cos^2(\theta - \alpha_i) = \sum_{i=1}^{6} k_i \left(\cos\theta \cos\alpha_i + \sin\theta \sin\alpha_i\right)^2$$

$$= \sum_{i=1}^{6} k_i \left(\cos^2\alpha_i \cos^2\theta + \sin^2\alpha_i \sin^2\theta\right)$$

$$+ 2 \sum_{i=1}^{6} \left(\cos\alpha_i \sin\alpha_i \cos\theta \sin\theta\right)$$

Natural frequency $= \omega_n = \sqrt{\dfrac{k_{eq}}{m}}$

For ω_n to be independent of θ, $\displaystyle\sum_{i=1}^{6} k_i \cos^2\alpha_i = \sum_{i=1}^{6} k_i \sin^2\alpha_i$ ...(E_1)

and $\displaystyle\sum_{i=1}^{6} k_i \cos\alpha_i \sin\alpha_i = 0$ ...(E_2)

(E_1) and (E_2) can be rewritten as

$$\sum_{i=1}^{6} k_i \left(\frac{1}{2} + \frac{1}{2}\cos 2\alpha_i\right) = \sum_{i=1}^{6} k_i \left(\frac{1}{2} - \frac{1}{2}\cos 2\alpha_i\right)$$

and $\displaystyle\frac{1}{2}\sum_{i=1}^{6} k_i \sin 2\alpha_i = 0$

i.e. $\displaystyle\sum_{i=1}^{6} k_i \cos 2\alpha_i = 0$ ... (E_3)

and $\displaystyle\sum_{i=1}^{6} k_i \sin 2\alpha_i = 0$ ... (E_4)

In the present example, (E_3) and (E_4) become

$$k_1 \cos 60^\circ + k_2 \cos 240^\circ + k_3 \cos 2\alpha_3 + k_1 \cos 420^\circ + k_2 \cos 600^\circ + k_3 \cos(360^\circ + 2\alpha_3) = 0$$

$$k_1 \sin 60^\circ + k_2 \sin 240^\circ + k_3 \sin 2\alpha_3 + k_1 \sin 420^\circ + k_2 \sin 600^\circ + k_3 \sin(360^\circ + 2\alpha_3) = 0$$

i.e., $\left.\begin{aligned} k_1 - k_2 + 2k_3 \cos 2\alpha_3 &= 0 \\ \sqrt{3}\, k_1 - \sqrt{3}\, k_2 + 2k_3 \sin 2\alpha_3 &= 0 \end{aligned}\right\}$; $\begin{aligned} 2k_3 \cos 2\alpha_3 &= k_2 - k_1 \quad \ldots (E_5) \\ 2k_3 \sin 2\alpha_3 &= \sqrt{3}\,(k_2 - k_1) \ldots (E_6) \end{aligned}$

Squaring (E_5) and (E_6) and adding,

$$4k_3^2 = (k_2 - k_1)^2 (1 + 3)$$

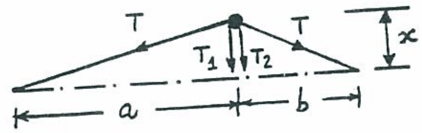
$$\therefore k_3 = \pm (k_2 - k_1) \Rightarrow k_3 = |k_2 - k_1|$$

Dividing (E_6) by (E_5),

$$\tan 2\alpha_3 = \sqrt{3}$$

$$\therefore \alpha_3 = \frac{1}{2} \tan^{-1}(\sqrt{3}) = 30^\circ$$

2.26 $T_1 = \frac{x}{a}\, T, \quad T_2 = \frac{x}{b}\, T$



(a) $m\ddot{x} + (T_1 + T_2) = 0$

$$m\ddot{x} + \left(\frac{T}{a} + \frac{T}{b}\right) x = 0$$

(b) $\omega_n = \sqrt{\dfrac{\frac{T}{a} + \frac{T}{b}}{m}} = \sqrt{\dfrac{T}{m\,a\,b}(a+b)}$

2.27 $m = \dfrac{160}{386.4}\ \dfrac{\text{lb—sec}^2}{\text{inch}}$, $k = 10$ lb/inch.

Velocity of jumper as he falls through 200 ft:

$$m\,g\,h = \frac{1}{2}\, m\, v^2 \quad \text{or} \quad v = \sqrt{2\,g\,h} = \sqrt{2\,(386.4)\,(200\,(12))} = 1{,}361.8811 \text{ in/sec}$$

About static equilibrium position:

$$x_0 = x(t{=}0) = 0, \quad \dot{x}_0 = \dot{x}(t{=}0) = 1{,}361.8811 \text{ in/sec}$$

Response of jumper: $\qquad x(t) = A_0 \sin(\omega_n t + \phi_0)$

where

$$A_0 = \left\{ x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2 \right\}^{\frac{1}{2}} = \frac{\dot{x}_0}{\omega_n} = \frac{\dot{x}_0 \sqrt{m}}{\sqrt{k}} = \frac{1361.8811}{\sqrt{10}} \sqrt{\left(\frac{160}{386.4}\right)} = 277.1281 \text{ in}$$

and $\qquad \phi_0 = \tan^{-1}\left(\dfrac{x_0\, \omega_n}{\dot{x}_0}\right) = 0$

2.28 The natural frequency of a vibrating rope is given by (see Problem 2.26):

$$\omega_n = \sqrt{\frac{T\,(a+b)}{m\,a\,b}}$$

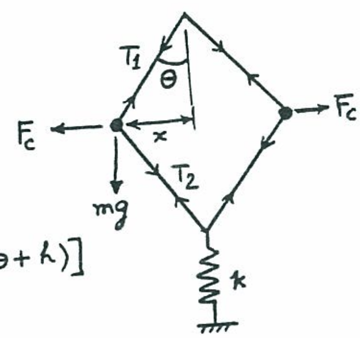
where T = tension in rope, m = mass, and a and b are lengths of the rope on both sides of the mass. For the given data:

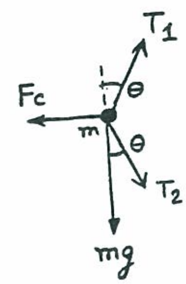
$$10 = \left\{ \frac{T\,(80+160)}{\left(\frac{120}{386.4}\right)(80)\,(160)} \right\}^{\frac{1}{2}} = \sqrt{T\,(0.060375)}$$

which yields

$$T = \frac{100}{0.060375} = 1{,}656.3147 \text{ lb}$$

2.29 When $\omega = 0$, total vertical height $= 2l + h$





When $\omega \neq 0$, total vertical height $= (2l\cos\theta + h)$

$F_c = m\,x\,\omega^2$

$x = l\sin\theta$

spring force $= k[2l + h - (2l\cos\theta + h)]$

$= 2kl\,(1 - \cos\theta)$

For vertical equilibrium of mass m,

$mg + T_2\cos\theta = T_1\cos\theta$ --- (E_1)

For horizontal equilibrium, $F_c = (T_1 + T_2)\sin\theta$

$T_2 = (F_c - T_1\sin\theta)/\sin\theta$ --- (E_2)

From (E_2), (E_1) can be expressed as

$$mg + \left(\frac{F_c - T_1\sin\theta}{\sin\theta}\right)\cos\theta = T_1\cos\theta$$

i.e. $$T_1 = \frac{mg + F_c\cot\theta}{2\cos\theta} = \frac{mg + m\omega^2 l\cos\theta}{2\cos\theta}$$

$$T_2 = \frac{F_c - T_1\sin\theta}{\sin\theta} = \frac{m\,x\,\omega^2 - \frac{mg}{2}\tan\theta - \frac{m\omega^2 l}{2}\sin\theta}{\sin\theta}$$

$$= \frac{m\,l\,\omega^2}{2} - \frac{mg}{2\cos\theta}$$

spring force $= 2kl\,(1 - \cos\theta) = 2T_2\cos\theta$

$= m\,l\,\omega^2\cos\theta - mg$

$$\cos\theta = \left(\frac{2kl + mg}{2kl + ml\omega^2}\right) \quad \text{---} \quad (E_3)$$

This equation defines the equilibrium position of mass m.
For small oscillations about the equilibrium position,
Newton's second law gives

$$2m\ddot{y} + ky = 0 \quad , \qquad \omega_n = \sqrt{\frac{2k}{m}}$$

2.30 (a) Let P = total spring force, F = centrifugal force acting on each ball. Equilibrium of moments about the pivot of bell crank lever (O) gives:

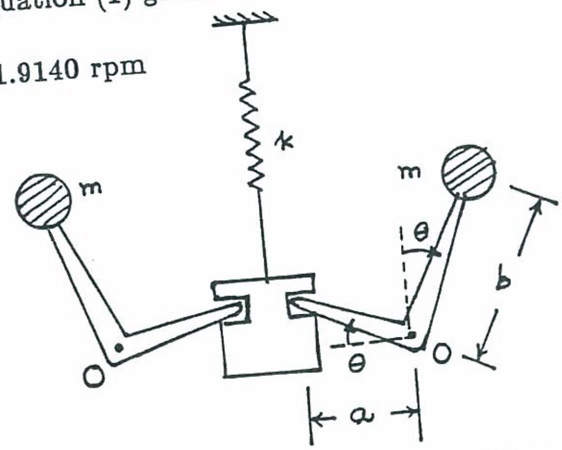
$$F\left(\frac{20}{100}\right) = \frac{P}{2}\left(\frac{12}{100}\right) \qquad (1)$$

When $P = 10^4\left(\frac{1}{100}\right) = 100$ N, and

$$F = m\,r\,\omega^2 = m\,r\left(\frac{2\pi N}{60}\right)^2 = \frac{25}{9.81}\left(\frac{16}{100}\right)\left(\frac{2\pi N}{60}\right)^2 = 0.004471\ N^2$$

where N = speed of the governor in rpm. Equation (1) gives:

$$0.004471\ N^2\ (0.2) = \frac{100}{2}\ (0.12) \quad \text{or} \quad N = 81.9140 \text{ rpm}$$



(b) Consider a small displacement of the ball arm about the vertical position. Equilibrium about point O gives:

$$(m\,b^2)\,\ddot{\theta} + (k\,a\sin\theta)\,a\cos\theta = 0 \qquad (2)$$

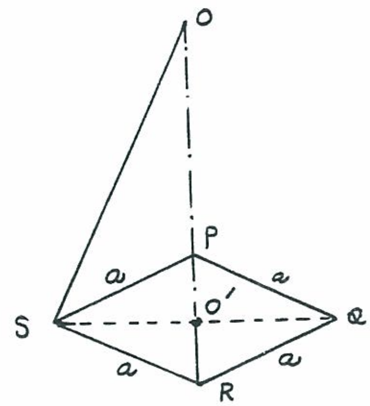
For small vallues of θ, $\sin\theta \approx \theta$ and $\cos\theta \approx 1$, and hence Eq. (2) gives

$$m\,b^2\,\ddot{\theta} + k\,a^2\,\theta = 0$$

from which the natural frequency can be determined as

$$\omega_n = \left\{\frac{k\,a^2}{m\,b^2}\right\}^{\frac{1}{2}} = \left\{(10)^4\left(\frac{0.12}{0.20}\right)^2\frac{9.81}{25}\right\}^{\frac{1}{2}} = 37.5851 \text{ rad/sec}$$

2.31 $SO' = \dfrac{a}{\sqrt{2}}$, $OO' = h$, $OS = \sqrt{h^2 + \dfrac{a^2}{2}}$



When each wire stretches by x_s, let the resulting vertical displacement of the platform be x.

$$OS + x_s = \sqrt{(h+x)^2 + \frac{a^2}{2}}$$

$$x_s = \sqrt{h^2 + \frac{a^2}{2}} \left\{ \sqrt{\frac{(h+x)^2 + \frac{a^2}{2}}{h^2 + \frac{a^2}{2}}} - 1 \right\}$$

$$= \sqrt{h^2 + \frac{a^2}{2}} \left[\sqrt{1 + \left\{ \frac{2hx + x^2}{\left(h^2 + \frac{a^2}{2}\right)} \right\}} - 1 \right]$$

For small x, x^2 is negligible compared to $2hx$ and $\sqrt{1+\theta} \simeq 1 + \dfrac{\theta}{2}$ and hence

$$x_s = \sqrt{h^2 + \frac{a^2}{2}} \left[1 + \frac{hx}{\left(h^2 + \frac{a^2}{2}\right)} - 1 \right] = \frac{h}{\sqrt{h^2 + \frac{a^2}{2}}} \cdot x$$

Potential energy equivalence gives

$$\frac{1}{2} k_{eq} x^2 = 4 \left(\frac{1}{2} k \, x_s^2 \right)$$

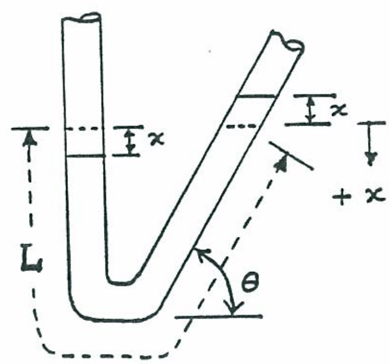
$$k_{eq} = 4k \left(\frac{x_s}{x} \right)^2 = \frac{4kh^2}{\left(h^2 + \frac{a^2}{2}\right)}$$

Equation of motion of M:

$$M \ddot{x} + k_{eq} x = 0$$

$$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{(k_{eq}/M)^{1/2}} = \frac{\pi \sqrt{M}}{h} \left(\frac{2h^2 + a^2}{2k} \right)^{1/2}$$

2.32 Equation of motion:



$$m \ddot{x} = \sum F_x$$

i.e., $(L A \rho) \ddot{x} = -2 (A x \rho g)$

i.e., $\ddot{x} + \dfrac{2g}{L} x = 0$

where A = cross-sectional area of the tube and ρ = density of mercury. Thus the natural frequency is given by:

$$\omega_n = \sqrt{\frac{2g}{L}}$$

2.33 Assume same area of cross section for all segments of the cable. Speed of blades = 300 rpm = 5 Hz = 31.416 rad/sec.

$$\omega_n^2 = \frac{k_{eq}}{m} = (2\ (31.416))^2 = (62.832)^2 \qquad (1)$$

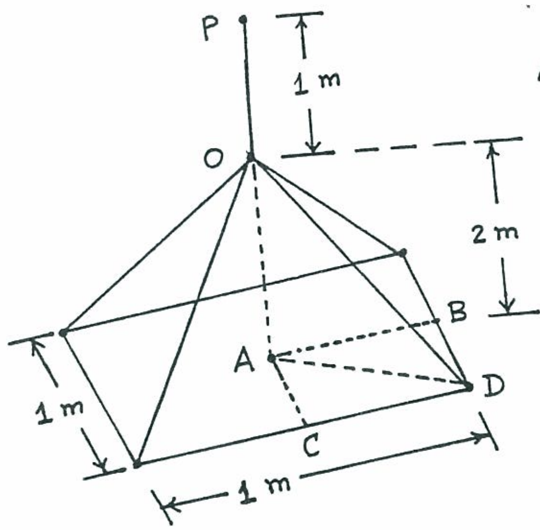
$$k_{eq} = m\ \omega_n^2 = 250\ (62.832)^2 = 98.6965\ (10^4)\ \text{N/m}$$

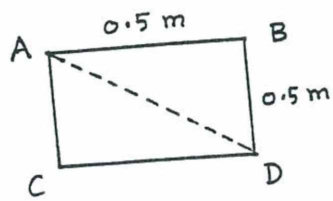
$$AD = \sqrt{0.5^2 + 0.5^2} = 0.7071\ \text{m}\ , \quad OD = \sqrt{2^2 + 0.7071^2} = 2.1213\ \text{m}$$

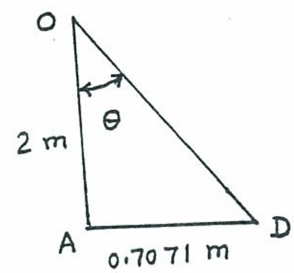
Stiffness of cable segments:

$$k_{PO} = \frac{A\,E}{\ell_{PO}} = \frac{A\ (207)\ (10^9)}{1} = 207\ (10^9)\ A\ \text{N/m}$$

$$K_{OD} = \frac{A\,E}{\ell_{OD}} = \frac{A\ (207)\ (10^9)}{2.1213} = 97.5817\ (10^9)\ A\ \text{N/m}$$







$$\theta = \tan^{-1}\left(\frac{0.7071}{2}\right)$$
$$= 19.4710°$$

The total sttiffness of the four inclined cables (k_{ic}) is given by:

$$k_{ic} = 4\ k_{OD}\ \cos^2\theta$$
$$= 4\ (97.5817)\ (10^9)\ A\ \cos^2\ 19.4710° = 346.9581\ (10^9)\ A\ \text{N/m}$$

Equivalent stiffness of vertical and inclined cables is given by:

$$\frac{1}{k_{eq}} = \frac{1}{k_{PO}} + \frac{1}{k_{ic}}$$

$$\text{i.e.,}\quad k_{eq} = \frac{k_{PO}\ k_{ic}}{k_{PO} + k_{ic}} \qquad (2)$$

$$= \frac{(207\ (10^9)\ A)\ (346.9581\ (10^9)\ A)}{(207\ (10^9)\ A) + (346.9581\ (10^9)\ A)} = 129.6494\ (10^9)\ A\ \text{N/m}$$

Equating k_{eq} given by Eqs. (1) and (2), we obtain the area of cross section of cables as:

$$A = \frac{98.6965\ (10^4)}{129.6494\ (10^9)} = 7.6126\ (10^{-6})\ \text{m}^2$$

2.34

$$\frac{1}{2\pi}\left\{\frac{k_1}{m}\right\}^{\frac{1}{2}} = 5 \; ; \quad \frac{k_1}{m} = 4\,(\pi)^2\,(25) = 986.9651$$

$$\frac{1}{2\pi}\left\{\frac{k_1}{m+5000}\right\}^{\frac{1}{2}} = 4.0825 \; ; \quad \frac{k_1}{m+5000} = 4\,(\pi)^2\,(16.6668) = 657.9822$$

Using $k_1 = \frac{A\,E}{\ell_1}$ we obtain

$$\frac{k_1}{m} = \frac{A\,E}{\ell_1\,m} = \frac{A\,(207)\,(10^9)}{2\,m} = 986.9651 \tag{1}$$

$$\text{i.e.,} \quad A = 9.5359\,(10^{-9})\,m$$

Also

$$\frac{k_1}{m+5000} = \frac{A\,E}{\ell_1\,(m+5000)} = 657.9822 \tag{2}$$

$$\text{i.e.,} \quad \frac{A}{m+5000} = 6.3573\,(10^{-9})$$

Using Eqs. (1) and (2), we obtain

$$A = 9.5359\,(10^{-9})\,m = 6.3573\,(10^{-9})\,m + 31.7865\,(10^{-6})$$

$$\text{i.e.,} \quad 3.1786\,(10^{-9})\,m = 31.7865\,(10^{-6}) \tag{3}$$

$$\text{i.e.,} \quad m = 10000.1573 \text{ kg}$$

Equations (1) and (3) yield

$$A = 9.5359\,(10^{-9})\,m = 9.5359\,(10^{-9})\,(10000.1573) = 0.9536\,(10^{-4})\,m^2$$

Longitudinal Vibration:

2.35

Let W_1 = part of weight W carried by length a of shaft

$W_2 = W - W_1$ = weight carried by length b

x = Elongation of length $a = \frac{W_1\,a}{AE}$

y = shortening of length $b = \frac{(W-W_1)(l-a)}{AE}$

E = Young's modulus

A = area of cross-section $= \pi d^2/4$

Since $x = y$, $\quad W_1 = \frac{W(l-a)}{l}$

x = elongation or static deflection of length $a = \frac{Wa\,(l-a)}{AEl}$

Considering the shaft of length a with end mass W_1/g as a spring-mass system,

$$\omega_n = \sqrt{\frac{g}{x}} = \left(\frac{g\,l\,AE}{Wa\,(l-a)}\right)^{1/2}$$

Transverse vibration:

spring constant of a fixed–fixed beam with off-center load

$$= k = \frac{3 E I \ell^3}{a^3 b^3} = \frac{3 E I \ell^3}{a^3 (\ell - a)^3}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \left\{ \frac{3 E I \ell^3 g}{W a^3 (\ell - a)^3} \right\}^{1/2} \quad \text{with } I = (\pi d^4 / 64) = \text{moment of inertia}$$

Torsional vibration:

If flywheel is given an angular deflection θ, resisting torques offered by lengths a and b are $\frac{G J \theta}{a}$ and $\frac{G J \theta}{b}$.

Total resisting torque $= M_t = G J \left(\frac{1}{a} + \frac{1}{b} \right) \theta$

$$k_t = \frac{M_t}{\theta} = G J \left(\frac{1}{a} + \frac{1}{b} \right) \quad \text{where } J = \frac{\pi d^4}{32} = \text{polar moment of inertia}$$

$$\omega_n = \sqrt{\frac{k_t}{J_0}} = \left\{ \frac{G J}{J_0} \left(\frac{1}{a} + \frac{1}{b} \right) \right\}^{1/2}$$

where J_0 = mass polar moment of inertia of the flywheel.

2.36 $m_{eq_{end}}$ = equivalent mass of a uniform beam at the free end (see Problem 2.38) =

$$\frac{33}{140} m = \frac{33}{140} \left\{ 1 \, (1) \, (150 \times 12) \, \frac{0.283}{386.4} \right\} = 0.3107$$

Stiffness of tower (beam) at free end:

$$k_b = \frac{3 E I}{L^3} = \frac{3 \, (30 \times 10^6) \, (\frac{1}{12} \, (1) \, (1^3))}{(150 \times 12)^3} = 0.001286 \text{ lb/in}$$

Length of each cable:

$OA = \sqrt{2} = 1.4142$ ft , $OB = \sqrt{2} \; 15 = 21.2132$ ft , $AB = OB - OA = 19.7990$ ft

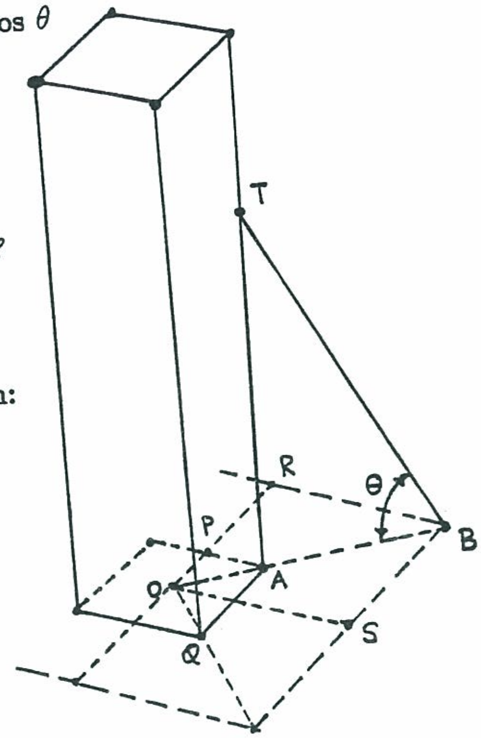
$$TB = \sqrt{TA^2 + AB^2} = \sqrt{100^2 + 19.7990^2} = 101.9412 \text{ ft}$$

$$\tan \theta = \frac{AT}{AB} = \frac{100}{19.7990} = 5.0508 \; , \quad \theta = 78.8008^\circ$$

Axial stiffness of each cable:

$$k = \frac{A E}{\ell} = \frac{(0.5) \, (30 \times 10^6)}{(101.9412 \times 12)} = 12261.971 \text{ lb/in}$$

Axial extension of each cable (y_c) due to a horizontal displacement of x of tower:



$$\ell_1^2 = \ell^2 + x^2 - 2\,\ell\,x\cos(180^\circ - \theta) = \ell^2 + x^2 + 2\,\ell\,x\cos\theta$$

$$\text{or}\quad \ell_1 = \ell\left\{1 + \left(\frac{x}{\ell}\right)^2 + 2\,\frac{x}{\ell}\cos\theta\right\}^{\frac{1}{2}}$$

$$y_c = \ell_1 - \ell \approx \ell\left(1 + \frac{1}{2}\,\frac{x^2}{\ell^2} + \frac{1}{2}\,(2)\,\frac{x}{\ell}\cos\theta\right) - \ell$$
$$= \ell + x\cos\theta - \ell = x\cos\theta$$

Equivalent stiffness of each cable in horizontal direction:

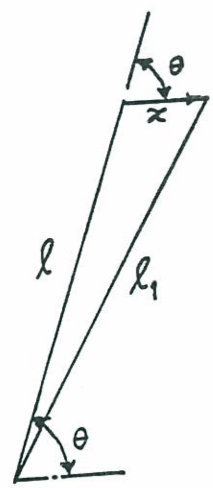
$$\frac{1}{2}\,k\,y_c^2 = \frac{1}{2}\,k_{eq_c}\,x^2 \quad \text{or} \quad k_{eq_c} = k\left(\frac{y_c}{x}\right)^2 = k\cos^2\theta$$

This gives

$$k_{eq_c} = (12261.971)\cos^2 78.8008^\circ = 462.5419\ \text{lb/in}$$

In order to use the relation

$$k_{eq_{end}} = k_b + 4\,k_{eq_c}\left(\frac{y_{L1}}{y_L}\right)^2$$



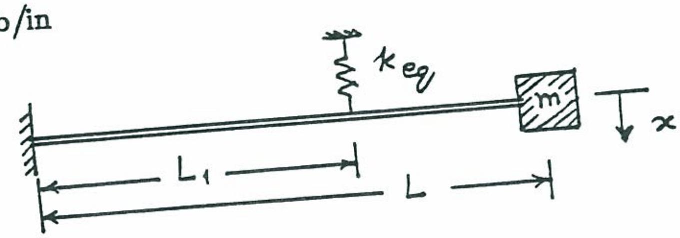
we find

$$\frac{y_{L1}}{y_L} = \left(\frac{F\,L_1^2\,(3\,L - L_1)}{6\,E\,I}\;\frac{3\,E\,I}{F\,L^3}\right) = \frac{L_1^2\,(3\,L - L_1)}{2\,L^3}$$
$$= \frac{100^2\,(3\,(150) - 100)}{2\,(150)^3} = 0.5185$$

Thus

$$k_{eq_{end}} = k_b + 4\,k_{eq_c}\,(0.5185)^2 = 0.001286 + 4\,(462.5419)\,(0.5185)^2$$
$$= 497.4045\ \text{lb/in}$$

Natural frequency:



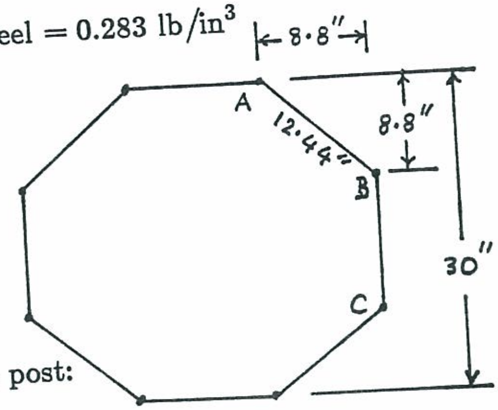
$$\omega_n = \left\{\frac{k_{eq_{end}}}{m_{eq_{end}}}\right\}^{\frac{1}{2}} = \left\{\frac{497.4045}{0.3107}\right\}^{\frac{1}{2}} = 40.0114\ \text{rad/sec}$$

2.37 Sides of the sign:

$$AB = \sqrt{8.8^2 + 8.8^2} = 12.44 \text{ in} \;;\; BC = 30 - 8.8 - 8.8 = 12.4 \text{ in}$$

$$\text{Area} = 30\,(30) - 4\,(\frac{1}{2}\,(8.8)\,(8.8)) = 745.12 \text{ in}^2$$

$$\text{Thickness} = \frac{1}{8} \text{ in} \;;\; \text{Weight density of steel} = 0.283 \text{ lb/in}^3$$



$$\text{Weight of sign} = (0.283)(\frac{1}{8})(745.12) = 26.64 \text{ lb}$$

$$\text{Weight of sign post} = (72)\,(2)\,(\frac{1}{4})\,(0.283) = 10.19 \text{ lb}$$

Stiffness of sign post (cantilever beam):

$$k = \frac{3\,E\,I}{\ell^3}$$

Area moments of inertia of the cross section of the sign post:

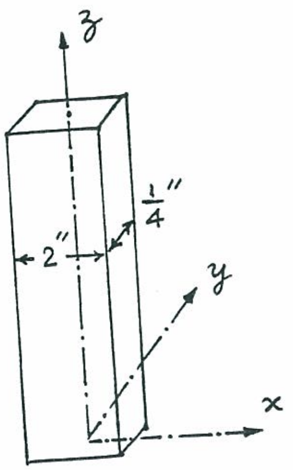
$$I_{xx} = \frac{1}{12}\,(2)\,(\frac{1}{4})^3 = \frac{1}{384} \text{ in}^4$$

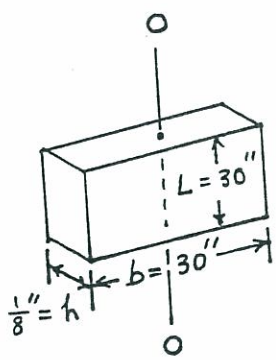
$$I_{yy} = \frac{1}{12}\,(\frac{1}{4})\,(2)^3 = \frac{1}{6} \text{ in}^4$$

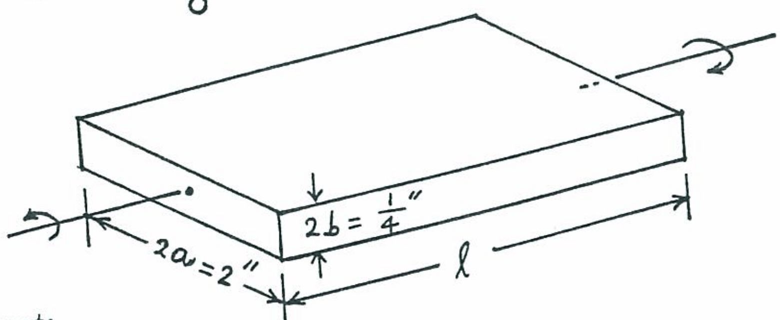
Bending stiffnesses of the sign post:

$$k_{xz} = \frac{3\,E\,I_{yy}}{\ell^3} = \frac{3\,(30\,(10^6))(\frac{1}{6})}{72^3} = 40.1877 \text{ lb/in}$$

$$k_{yz} = \frac{3\,E\,I_{xx}}{\ell^3} = \frac{3\,(30\,(10^6))\,(\frac{1}{384})}{72^3} = 0.6279 \text{ lb/in}$$







Torsional stiffness of the sign post:

$$k_t = 5.33\ \frac{a\ b^3}{\ell}\ G \left\{1 - 0.63\ \frac{b}{a} \left(1 - \frac{b^4}{12\ a^4}\right)\right\}$$

(See Ref: N. H. Cook, *Mechanics of Materials for Design*, McGraw-Hill, New York, 1984, p. 342).

Thus

$$k_t = 5.33 \left\{\frac{(1)\ (\frac{1}{8})^3}{72}\right\} (11.5\ (10^6)) \left\{1 - (0.63)\ (\frac{1}{8}) \left(1 - \frac{(\frac{1}{8})^4}{12\ (1)^4}\right)\right\}$$

$$= 1531.7938 \text{ lb—in/rad}$$

Natural frequency for bending in xz plane:

$$\omega_{xz} = \left\{\frac{k_{xz}}{m}\right\}^{\frac{1}{2}} = \left\{\frac{40.1877}{\left(\frac{26.64}{386.4}\right)}\right\}^{\frac{1}{2}} = 24.1434 \text{ rad/sec}$$

Natural frequency for bending in yz plane:

$$\omega_{yz} = \left\{\frac{k_{yz}}{m}\right\} = \left\{\frac{0.6279}{\left(\frac{26.64}{386.4}\right)}\right\}^{\frac{1}{2}} = 3.0178 \text{ rad/sec}$$

By approximating the shape of the sign as a square of side 30 in (instead of an octagon), we can find its mass moment of inertia as:

$$I_{oo} = \frac{\gamma L}{3}\ (b^3\ h + h^3\ b) = \left(\frac{0.283}{386.4}\right) \left(\frac{30}{3}\right) \left(30^3\ (\frac{1}{8}) + (\frac{1}{8})^3\ (30)\right) = 24.7189$$

Natural torsional frequency:

$$\omega_t = \left\{\frac{k_t}{I_{oo}}\right\}^{\frac{1}{2}} = \left\{\frac{1531.7938}{24.7189}\right\}^{\frac{1}{2}} = 7.8720 \text{ rad/sec}$$

Thus the mode of vibration (close to resonance) is torsion in xy plane.

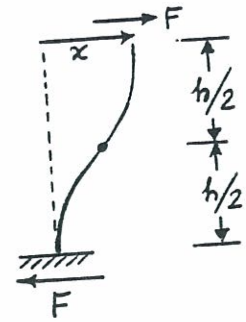
2.38 (a) Pivoted: Let $l = h$.

$$k_{eq} = 4\ k_{column} = 4 \left(\frac{3EI}{l^3}\right) = \frac{12EI}{l^3}$$

Let $m_{eff\,1}$ = effective mass due to self weight of columns

Equation of motion: $\left(\frac{W}{g} + m_{eff\,1}\right) \ddot{x} + k_{eq}\, x = 0$

Natural frequency of horizontal vibration $= \omega_n = \sqrt{\dfrac{12EI}{l^3\left(\frac{W}{g} + m_{eff\,1}\right)}}$



(b) Fixed:

Since the joint between column and floor does not permit rotation, each column will bend with inflection point at middle.

When force F is applied at ends,

$$x = 2 \; \frac{F\,(l/2)^3}{3\,EI} = \frac{F l^3}{12\,EI}$$

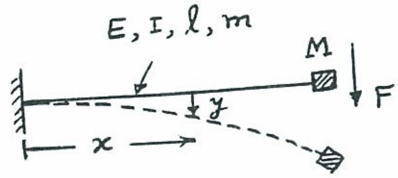
$$k_{column} = \frac{12\,EI}{l^3} \quad ; \quad k_{eq} = 4\, k_{column} = \frac{48\,EI}{l^3}$$

Let $m_{eff\,2}$ = effective mass of each column at top end

Equation of motion: $\left(\frac{W}{g} + m_{eff\,2}\right) \ddot{x} + k_{eq}\, x = 0$

Natural frequency of horizontal vibration $= \omega_n = \sqrt{\dfrac{48\,EI}{l^3\left(\frac{W}{g} + m_{eff\,2}\right)}}$

Effective mass (due to self weight):



(a) Let $m_{eff\,1}$ = effective part of mass of beam (m) at end.

Thus vibrating inertia force at end is due to $(M + m_{eff\,1})$.

Assume deflection shape during vibration same as the static deflection shape with a tip load:

$y(x,t) = Y(x)\,\cos(\omega_n t - \phi)$ where $Y(x) = \dfrac{F x^2\,(3l - x)}{6EI}$.

$Y(x) = \dfrac{Y_0}{2 l^3}\, x^2\,(3l - x)$ where $Y_0 = \dfrac{F l^3}{3EI}$ = max. tip deflection

$$y(x,t) = \frac{Y_0}{2 l^3}\,(3 x^2 l - x^3)\,\cos(\omega_n t - \phi) \qquad (E_1)$$

Max. strain energy of beam = Max. work by force F

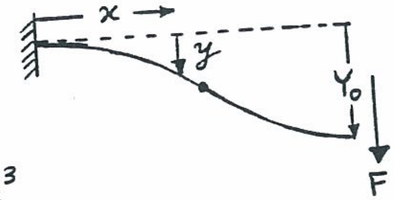
$$= \frac{1}{2} F Y_0 = \frac{3}{2}\,\frac{EI}{l^3}\, Y_0^2 \qquad (E_2)$$

Max. kinetic energy due to distributed mass of beam

$$= \frac{1}{2}\,\frac{m}{l} \int_0^l \dot{y}^2(x,t)\Big|_{max}\, dx + \frac{1}{2}\left(\dot{y}_{max}\right)^2 M$$

$$= \frac{1}{2}\,\omega_n^2\, Y_0^2 \left(\frac{33}{140}\, m\right) + \frac{1}{2}\,\omega_n^2\, Y_0^2\, M \qquad (E_3)$$

$$\therefore \; m_{eff\,1} = \frac{33}{140}\, m = 0.2357\, m$$



(b) Let $Y(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3$

$Y(0) = 0, \quad \frac{dY}{dx}(0) = 0, \quad Y(l) = Y_0, \quad \frac{dY}{dx}(l) = 0$

This leads to $\quad Y(x) = \frac{3 Y_0}{l^2} x^2 - \frac{2 Y_0}{l^3} x^3$ (E4)

$$y(x,t) = Y_0 \left(3 \frac{x^2}{l^2} - 2 \frac{x^3}{l^3} \right) \cos(\omega_n t - \phi)$$

Maximum strain energy $= \frac{1}{2} E I \int_0^l \left(\frac{\partial^2 y}{\partial x^2} \right)^2 dx \Big|_{max}$ (E5)

$$= \frac{6 E I Y_0^2}{l^3}$$

Max. kinetic energy $= \frac{1}{2} M \omega_n^2 Y_0^2 + \frac{1}{2} \left(\frac{m}{l} \right) Y_0^2 \omega_n^2 \int_0^l \left(\frac{3x^2}{l^2} - \frac{2x^3}{l^3} \right)^2 dx$ (E6)

$$= \frac{1}{2} \omega_n^2 Y_0^2 \left(M + \frac{13}{35} m \right)$$

$$\therefore \; m_{eff\,2} = \frac{13}{35} m = 0.3714 \, m$$

2.39 Stiffnesses of segments:

$$A_1 = \frac{\pi}{4} (D_1^2 - d_1^2) = \frac{\pi}{4} (2^2 - 1.75^2) = 0.7363 \text{ in}^2$$

$$k_1 = \frac{A_1 E_1}{L_1} = \frac{(0.7363)(10^7)}{12} = 61.3583 \, (10^4) \text{ lb/in}$$

$$A_2 = \frac{\pi}{4} (D_2^2 - d_2^2) = \frac{\pi}{4} (1.5^2 - 1.25^2) = 0.5400 \text{ in}^2$$

$$k_2 = \frac{A_2 E_2}{L_2} = \frac{(0.5400)(10^7)}{10} = 54.0 \, (10^4) \text{ lb/in}$$

$$A_3 = \frac{\pi}{4} (D_3^3 - d_3^2) = \frac{\pi}{4} (1^2 - 0.75^2) = 0.3436 \text{ in}^2$$

$$k_3 = \frac{A_3 E_3}{L_3} = \frac{(0.3436)(10^7)}{8} = 42.9516 \, (10^4) \text{ lb/in}$$

Equivalent stiffness (springs in series):

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}$$

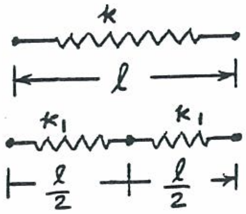
$$= 0.0162977 \, (10^{-4}) + 0.0185185 \, (10^{-4}) + 0.0232820 \, (10^{-4}) = 0.0580982 \, (10^{-4})$$

$$\text{or} \quad k_{eq} = 17.2122 \, (10^4) \text{ lb/in}$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k_{eq}\, g}{W}} = \sqrt{\frac{17.2122 \, (10^4)(386.4)}{10}} = 2578.9157 \text{ rad/sec}$$

2.40



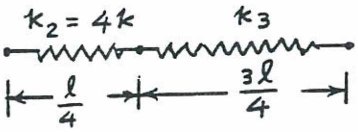
$$\frac{1}{k_{total}} = \frac{1}{k_1} + \frac{1}{k_1}$$

$$k_{total} = \frac{k_1}{2} \equiv k \; ; \; k_1 = 2k$$

$$\tau_n = 2\pi \sqrt{\frac{m}{k_{eq}}}$$

$$0.5 = 2\pi \sqrt{\frac{m}{4k}}$$

$$\sqrt{\frac{m}{k}} = \frac{1}{2\pi}$$



$$\frac{1}{k_{total}} = \frac{1}{k_2} + \frac{1}{k_3} = \frac{1}{4k} + \frac{1}{k_3} = \frac{1}{k}$$

$$k_3 = \frac{4}{3} k$$

$$\tau_n = 2\pi \sqrt{\frac{m}{k_{eq}}} \quad \text{where} \quad k_{eq} = 4k + \frac{4}{3}k = \frac{16}{3}k$$

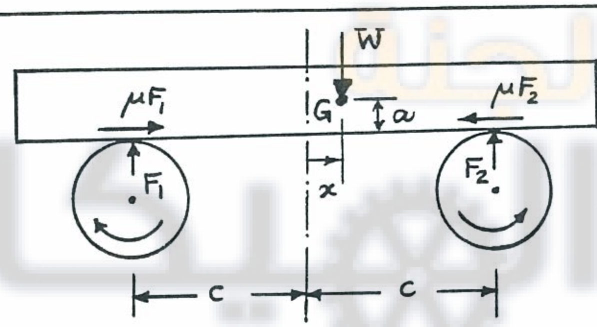
$$\therefore \tau_n = 2\pi \sqrt{\frac{3m}{16k}} = \frac{2\pi\sqrt{3}}{4}\sqrt{\frac{m}{k}} = \frac{2\pi\sqrt{3}}{4}\left(\frac{1}{2\pi}\right) = 0.4330 \text{ sec}$$

2.41

Let μ = coefficient of friction

x = displacement of C.G. of block

F_1, F_2 = net reactions between roller and block on left and right sides.



Reactions at left and right due to static load W are $W(c-x)/2c$ and $W(c+x)/2c$, respectively.

M = moment about G due to motion of block $= (\mu F_2 - \mu F_1)a$

Reactions at left and right to balance $M = \frac{M}{2c} = \frac{\mu a}{2c}(F_2 - F_1)$

$$F_1 = \frac{W(c-x)}{2c} - \frac{\mu a}{2c}(F_2 - F_1) \; ; \quad F_2 = \frac{W(c+x)}{2c} + \frac{\mu a}{2c}(F_2 - F_1)$$

subtraction gives $\quad F_2 - F_1 = \frac{Wx}{c} + \frac{\mu a}{c}(F_2 - F_1)$

i.e., $\quad F_2 - F_1 = \frac{Wx}{c}\left(\frac{c}{c-\mu a}\right) = \frac{Wx}{c-\mu a}$

Restoring force $= \mu(F_2 - F_1) = \left(\frac{\mu W x}{c-\mu a}\right)$

Equation of motion : $\quad \frac{W}{g}\ddot{x} + \frac{\mu W}{(c-\mu a)} x = 0$

$$\omega_n = \omega = \sqrt{\frac{\mu W g}{W(c-\mu a)}} = \sqrt{\frac{\mu g}{c-\mu a}}$$

Solving this, we get $\quad \omega = \left[c\,\omega^2/(g + a\,\omega^2)\right]$

2.42 From problem 2.41,

Restoring force without springs $= \mu (F_2 - F_1) = \frac{\mu W x}{c - \mu a}$

spring restoring force $= 2 k x$

Total restoring force $= \frac{\mu W x}{c - \mu a} + 2 k x$

Equation of motion: $\frac{W}{g} \ddot{x} + \left(\frac{\mu W}{c - \mu a} + 2 k \right) x = 0$

$$\omega_n = \omega = \left\{ \frac{[\mu W + 2 k (c - \mu a)] g}{(c - \mu a) W} \right\}^{1/2}$$

Solution of this equation gives

$$\mu = \left(\frac{\omega^2 W c - 2 k g c}{W g + W \omega^2 a - 2 k g a} \right)$$

2.43 (a) Natural frequency of vibration of electromagnet (without the automobile):

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{10000.0\ (386.4)}{3000.0}} = 35.8887 \text{ rad/sec}$$

(b) When the automobile is dropped, the electromagnet moves up by a distance (x_0) from its static equilibrium position.

$x_0 =$ static deflection (elongation of cable) under the weight of automobile

$$= \frac{W_{auto}}{k} = \frac{2000}{10000} = 0.2 \text{ in}$$

$\dot{x}_0 = 0$

Resultant motion of electromagnet (+x upwards):

$$x(t) = A_0 \sin (\omega_n t + \phi_0)$$

where

$$A_0 = \left\{ x_0^2 + \left(\frac{\dot{x}_0}{\omega_n} \right)^2 \right\}^{\frac{1}{2}} = x_0 = 0.2$$

and $$\phi_0 = \tan^{-1} \left(\frac{x_0\ \omega_n}{\dot{x}_0} \right) = \tan^{-1} (\infty) = 90^\circ$$

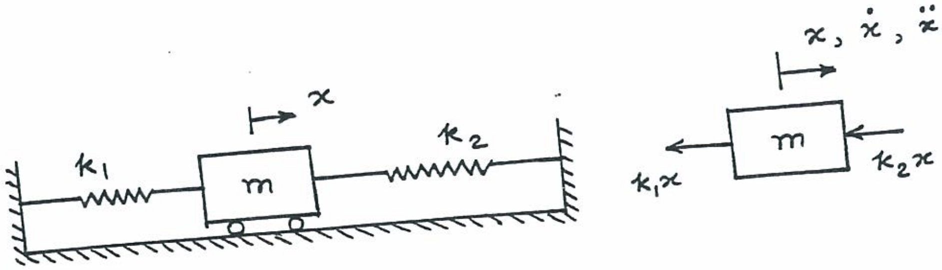
Hence $x(t) = 0.2 \sin (35.8887\ t + 90^\circ) = 0.2 \cos 35.8887\ t$

(c) Maximum x(t):

$$x(t)\,|_{max} = A_0 = 0.2 \text{ in}$$

Maximum tension in cable during motion $= k\, x(t)\,|_{max} +$ Weigh of electromagnet $= 10000\ (0.2) + 3000 = 5{,}000$ lb.

2.44



(a) Newton's second law of motion:

$$F(t) = -k_1\,x - k_2\,x = m\,\ddot{x} \quad \text{or} \quad m\,\ddot{x} + (k_1 + k_2)\,x = 0$$

(b) D'Alembert's principle:

$$F(t) - m\,\ddot{x} = 0 \quad \text{or} \quad -k_1\,x - k_2\,x - m\,\ddot{x} = 0$$

$$\text{Thus} \quad m\,\ddot{x} + (k_1 + k_2)\,x = 0$$

(c) Principle of virtual work:

When mass m is given a virtual displacement δx,
Virtual work done by the spring forces = - $(k_1 + k_2)\,x\,\delta x$
Virtual work done by the inertia force = - $(m\,\ddot{x})\,\delta x$
According to the principle of virtual work, the total virtual work done by all forces must be equal to zero:

$$-m\,\ddot{x}\,\delta x - (k_1 + k_2)\,x\,\delta x = 0 \quad \text{or} \quad m\,\ddot{x} + (k_1 + k_2)\,x = 0$$

(d) Principle of conservation of energy:

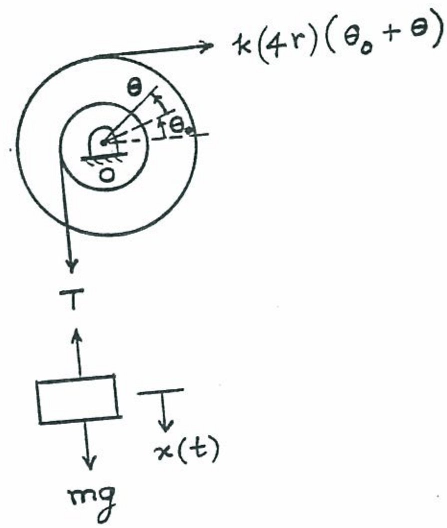
$$T = \text{kinetic energy} = \frac{1}{2}\,m\,\dot{x}^2$$

$$U = \text{strain energy} = \text{potential energy} = \frac{1}{2}\,k_1\,x^2 + \frac{1}{2}\,k_2\,x^2$$

$$T + U = \frac{1}{2}\,m\,\dot{x}^2 + \frac{1}{2}\,(k_1 + k_2)\,x^2 = c = \text{constant}$$

$$\frac{d}{dt}\,(T + U) = 0 \quad \text{or} \quad m\,\ddot{x} + (k_1 + k_2)\,x = 0$$

2.45



Equation of motion:

$$\text{Mass m:}\quad m\,g - T = m\,\ddot{x} \tag{1}$$

$$\text{Pulley } J_0:\quad J_0\,\ddot{\theta} = T\,r - k\,4\,r\,(\theta + \theta_0)\,4\,r \tag{2}$$

where θ_0 = angular deflection of the pulley under the weight, mg, given by:

$$m\,g\,r = k\,(4\,r\,\theta_0)\,4\,r \quad \text{or} \quad \theta_0 = \frac{m\,g}{16\,r\,k} \tag{3}$$

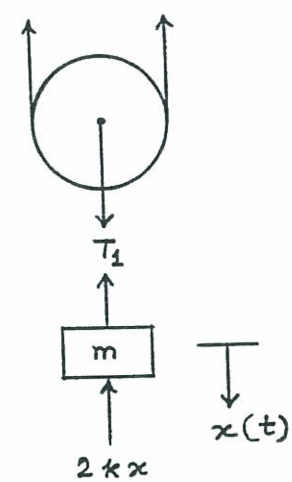
Substituting Eqs. (1) and (3) into (2), we obtain

$$J_0\,\ddot{\theta} = (m\,g - m\,\ddot{x})\,r - k\,16\,r^2\,(\theta + \frac{m\,g}{16\,r\,k}) \tag{4}$$

Using $x = r\,\theta$ and $\ddot{x} = r\,\ddot{\theta}$, Eq. (4) becomes

$$(J_0 + m\,r^2)\,\ddot{\theta} + (16\,r^2\,k)\,\theta = 0$$

2.46 Consider the springs connected to the pulleys (by rope) to be in series. Then



$$\frac{1}{k_{eq}} = \frac{1}{k} + \frac{1}{5k} \quad \text{or} \quad k_{eq} = \frac{5}{6}\,k$$

Let the displacement of mass m be x. Then the extension of the rope (springs connected to the pulleys) = 2 x. From the free body diagram, the equation of motion of mass m:

$$m\,\ddot{x} + 2\,k\,x + k_{eq}\,(2\,x) = 0$$

$$\text{or} \quad m\,\ddot{x} + \frac{11}{3}\,k\,x = 0$$

2.47 T = kinetic energy = $T_{mass} + T_{pulley}$

$$= \frac{1}{2}\,m\,\dot{x}^2 + \frac{1}{2}\,J_0\,\dot{\theta}^2 = \frac{1}{2}\,(m\,r^2 + J_0)\,\dot{\theta}^2$$

$$U = \text{potential energy} = \frac{1}{2}\,k\,x_s^2 = \frac{1}{2}\,k\,(4\,r\,\theta)^2 = \frac{1}{2}\,k\,(16\,r^2)\,\theta^2$$

Using $\frac{d}{dt}\,(T + U) = 0$ gives

$$(m\,r^2 + J_0)\,\ddot{\theta} + (16\,r^2\,k)\,\theta = 0$$

2.48 $T = \text{kinetic energy} = \frac{1}{2}\,m\,\dot{x}^2 + \frac{1}{2}\,J_0\,\dot{\theta}^2$

$$U = \text{potential energy} = \frac{1}{2}\,k\,x_s^2$$

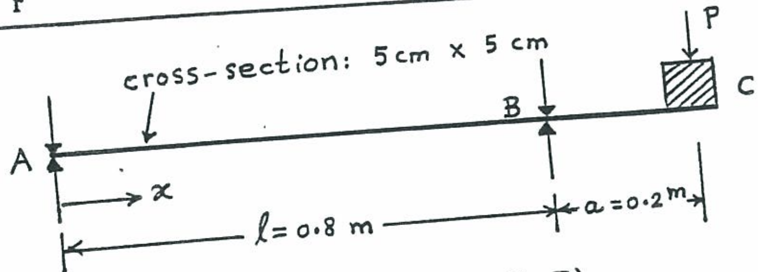
where $\theta = \frac{x}{r}$, x_s = extension of spring $= 4\, r\, \theta = 4\, x$. Hence

$$T = \frac{1}{2} \left(m + \frac{J_0}{r^2}\right) \dot{x}^2 \quad ; \quad U = \frac{1}{2} (16\, k)\, x^2$$

Using tthe relation $\frac{d}{dt} (T + U) = 0$, we obtain the equation of motion of the system as:

$$\left(m + \frac{J_0}{r^2}\right) \ddot{x} + 16\, k\, x = 0$$

2.49



Due to a load P at C, deflection at point C is given by (from Appendix B):

$$y(x) = \frac{P\,(x - \ell)}{6\, E\, I\, \ell} \left[a\, (3\, x - \ell) - (x - \ell)^2 \right] ; \ell \le x \le \ell + a$$

$$y_C = y(x = \ell + a) = \frac{P\, a^2}{3\, E\, I\, \ell}\, (\ell + a)$$

Moment of inertia of cross section of beam:

$$I = \frac{1}{12}\, (0.05)\, (0.05)^3 = 52.0833\, (10^{-8})\ m^4$$

Equivalent stiffness:

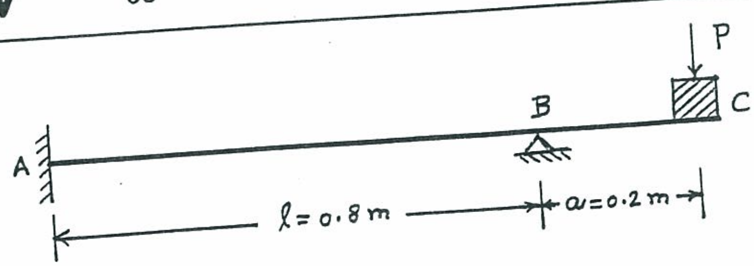
$$k_{eq} = \frac{P}{y_C} = \frac{3\, E\, I\, \ell}{a^2\, (\ell + a)} = \frac{3\, (207\, (10^9))\, (52.0833\, (10^{-8}))\, (0.8)}{(0.2)^2\, (0.8 + 0.2)}$$

$$= 6.4687\, (10^6)\ N/m$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{6.4687\, (10^6)}{50}} = 359.6872\ rad/sec$$

2.50



From Appendix B, the deflection of fixed-pinned beam with an overhang, due to load P at the free end, is given by:

$$y(x) = \frac{P\,a}{4\,E\,I\,\ell}\left[x^2 - \ell\,x^2 - \left(\frac{2\,\ell}{3\,a} + 1\right)(x-\ell)^3\right];\ \ell \le x \le \ell + a$$

Using $a = 0.2$, $\ell = 0.8$, $x = a + \ell = 1.0$, and

$$I = \frac{1}{12}\,(0.05)\,(0.05)^3 = 52.0833\,(10^{-8})\ m^4$$

we obtain

$$y_C = \frac{P\,(0.2)}{4\,(207\,(10^9))\,(52.0833\,(10^{-8}))\,(0.8)}\left[1^2 - 0.8\,(1)^2 - \left(\frac{1.6}{0.6}+1\right)(0.2)^3\right]$$

$$= P\,(9.895652\,(10^{-8}))$$

$$k_{eq} = \frac{P}{y_C} = 1010.5448\,(10^4)\ N/m$$

$$\omega_n = \left\{\frac{k_{eq}}{m}\right\}^{\frac{1}{2}} = \left\{\frac{10.1054\,(10^6)}{50}\right\}^{\frac{1}{2}} = 449.5642\ rad/sec$$

(2.51) Data: $k = 500$ N/m, $m = 2$ kg, $x_0 = 0.1$ m, $\dot{x}_0 = 5$ m/s

$$\omega_n = \sqrt{k/m} = (500/2)^{1/2} = 15.8114\ rad/s$$

Eq. (2.18): $x(t) = x_0 \cos\omega_n t + \dfrac{\dot{x}_0}{\omega_n}\sin\omega_n t$

$$= 0.1\cos 15.8114\,t + \left(\frac{5}{15.8114}\right)\sin 15.8114\,t$$

$$\therefore\ x(t) = 0.1\cos 15.8114\,t + 0.3162\sin 15.8114\,t\ \ m$$

$$\dot{x}(t) = -1.5811\sin 15.8114\,t + 5\cos 15.8114\,t\ \ m/s$$

$$\ddot{x}(t) = -24.9994\cos 15.8114\,t - 79.0570\sin 15.8114\,t\ \ \frac{m}{s^2}$$

(2.52) Data: $\omega_n = 10$ rad/s, $x_0 = 0.05$ m, $\dot{x}_0 = 1$ m/s

Response of undamped system:

$$x(t) = x_0\cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\sin\omega_n t$$

$$= 0.05\cos 10t + \left(\frac{1}{10}\right)\sin 10t$$

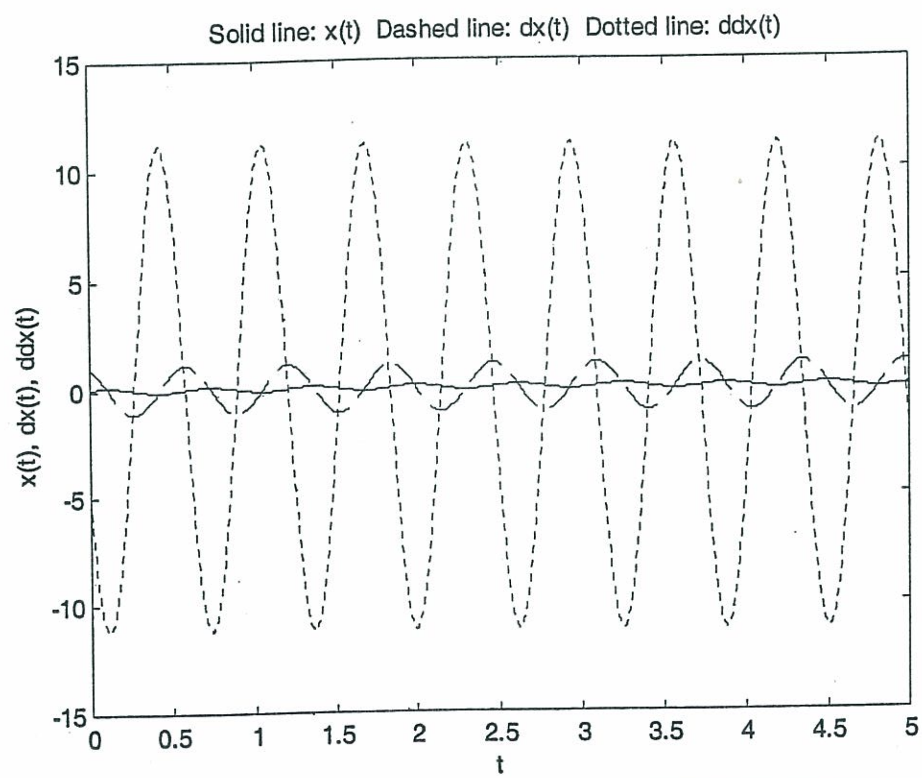
$$\therefore\ x(t) = 0.05 \cos 10t + 0.1 \sin 10t \quad m \tag{E.1}$$

$$\dot{x}(t) = -0.5 \sin 10t + \cos 10t \quad m/s \tag{E.2}$$

$$\ddot{x}(t) = -5 \cos 10t - 10 \sin 10t \quad m/s^2 \tag{E.3}$$

Plotting of Eqs. (E.1) to (E.3):

```
% Ex2_52.m
for i = 1: 1001
   t(i) = (i-1)*5/1000;
   x(i) = 0.05 * cos(10*t(i)) + 0.1*sin(10*t(i));
   dx(i) = -0.5*sin(10*t(i)) + cos(10*t(i));
   ddx(i) = -5*cos(10*t(i)) - 10*sin(10*t(i));
end
plot(t, x);
hold on;
plot(t, dx, '--');
plot(t, ddx, ':');
xlabel('t');
ylabel('x(t), dx(t), ddx(t)');
title('Solid line: x(t)  Dashed line: dx(t)  Dotted line: ddx(t)');
```



2.53 Data: $\omega_d = 2$ rad/s, $\zeta = 0.1$, $X_0 = 0.01$ m, $\phi_0 = 1$ rad

Initial conditions ?

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n, \quad \omega_n = \omega_d/\sqrt{1-\zeta^2} = 2/\sqrt{1-0.01} = 2.0101 \text{ rad/s} \qquad (E.1)$$

Eqs. (2.73), (2.75):

$$X_0 = \left\{ x_0^2 + \left(\frac{\dot{x}_0 + \zeta \omega_n x_0}{\omega_d} \right)^2 \right\}^{\frac{1}{2}} = 0.01 \qquad (E.2)$$

$$\phi_0 = \tan^{-1}\left(- \frac{\dot{x}_0 + \zeta \omega_n x_0}{\omega_d x_0} \right) = 1 \qquad (E.3)$$

Eqs. (E.2) and (E.3) lead to:

$$x_0^2 + \left(\frac{\dot{x}_0 + 0.20101\, x_0}{2} \right)^2 = 0.0001 \qquad (E.4)$$

$$-\left(\frac{\dot{x}_0 + 0.20101\, x_0}{2 x_0} \right) = \tan 1 = 0.7854$$

or $\quad -(\dot{x}_0 + 0.20101\, x_0) = 1.5708\, x_0 \qquad (E.5)$

Substitution of Eq. (E.5) into (E.4) yields

$$x_0 = 0.007864 \text{ m} \qquad (E.6)$$

Eqs. (E.6) and (E.5) give

$$\dot{x}_0 = -0.013933 \text{ m/s} \qquad (E.7)$$

2.54 Without passengers,

$$(\omega_n)_1 = \sqrt{\frac{k}{m}} = 20 \text{ rad/s} \Rightarrow k = 400\, m \qquad (E.1)$$

With passengers,

$$(\omega_n)_2 = \sqrt{\frac{k}{m+500}} = 17.32 \text{ rad/s} \qquad (E.2)$$

Squaring Eq. (E.2), we get

$$\frac{k}{m+500} = (17.32)^2 = 299.9824 \qquad (E.3)$$

Using $k = 400\, m$ in (E.3) gives

$$m = 1499.6481 \text{ kg}$$

2.55

$$\omega_n = \sqrt{k/m} = \sqrt{3200/2} = 40 \text{ rad/s}$$

$$x_0 = 0$$

$$X_0 = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2} = 0.1$$

i.e. $\frac{\dot{x}_0}{\omega_n} = 0.1$ or $\dot{x}_0 = 0.1\,\omega_n = 4$ m/s

---

2.56 Data: $D = 0.5625''$, $G = 11.5 \times 10^6$ psi, $\rho = 0.282$ lb/in^3

$f = 193$ Hz, $k = 26.4$ lb/in

$$k = \text{spring rate} = \frac{d^4 G}{8 D^3 N} \Rightarrow \frac{d^4 (11.5 \times 10^6)}{8 (0.5625^3) N} = 26.4$$

or $$\frac{d^4}{N} = \frac{26.4 (8)(0.5625^3)}{11.5 \times 10^6} = 3.2686 \times 10^{-6} \quad \text{(E.1)}$$

$$f = \frac{1}{2}\sqrt{\frac{k g}{W}}$$

where $W = \left(\frac{\pi d^2}{4}\right) \pi D N \rho = \frac{\pi^2}{4}(0.5625)(0.282) N d^2$

$= 0.391393 \ N d^2$

Hence $$f = \frac{1}{2}\sqrt{\frac{26.4 (386.4)}{0.391393 \ N d^2}} = 193$$

(E.2)

or $N d^2 = 0.174925$

Eqs. (E.1) and (E.2) yield

$$N = \frac{d^4}{3.2686 \times 10^{-6}} = \frac{0.174925}{d^2}$$

or $d^6 = 0.571764 \times 10^{-6}$

or $d = 0.911037 \times 10^{-1} = 0.0911037$ inch

Hence $$N = \frac{0.174925}{d^2} = 21.075641$$

2.57

Data: $D = 0.5625''$, $G = 4 \times 10^6$ psi, $\rho = 0.1$ lb/in^3

$f = 193$ Hz, $k = 26.4$ lb/in

$$k = \text{spring rate} = \frac{d^4 G}{8 D^3 N} \Rightarrow \frac{d^4 (4 \times 10^6)}{8 (0.5625^3) N} = 26.4$$

or
$$\frac{d^4}{N} = \frac{26.4 (8)(0.5625^3)}{4 \times 10^6} = 9.397266 \times 10^{-6} \quad (E.1)$$

$$f = \text{frequency} = \frac{1}{2} \sqrt{\frac{k g}{W}}$$

where
$$W = \left(\frac{\pi d^2}{4}\right) \pi D N \rho = \frac{\pi^2}{4} (0.5625)(0.1) N d^2$$
$$= 0.138792 \ N d^2$$

Hence
$$f = \frac{1}{2} \sqrt{\frac{26.4 (386.4)}{0.138792 \ N d^2}} = 193$$

(E.2)

or $N d^2 = 0.493290$

Eqs. (E.1) and (E.2) yield
$$N = \frac{d^4}{9.397266 \times 10^{-6}} = \frac{0.493290}{d^2}$$

or $d^6 = 4.635575 \times 10^{-6}$

or $d = 0.129127$ inch

Hence
$$N = \frac{0.493290}{d^2} = 29.584728$$

2.58

By neglecting the effect of self weight of the beam, and using a single degree of freedom model, the natural frequency of the system can be expressed as
$$\omega_n = \sqrt{\frac{k}{m}}$$

where m = mass of the machine, and
k = stiffness of the cantilever beam:

$$k = \frac{3EI}{l^3}$$

where l = length, E = Young's modulus, and I = area moment of inertia of the beam section. Assuming $E = 30 \times 10^6$ psi for steel and 10.5×10^6 psi for aluminum, we have

$$(\omega_n)_{steel} = \left\{ \frac{3\,(30 \times 10^6)\, I}{m\, l^3} \right\}^{\frac{1}{2}}$$

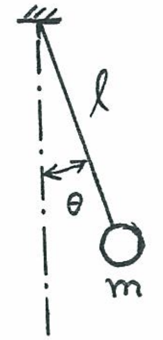
$$(\omega_n)_{aluminum} = \left\{ \frac{3\,(10.5 \times 10^6)\, I}{m\, l^3} \right\}^{\frac{1}{2}}$$

Ratio of natural frequencies:

$$\frac{(\omega_n)_{steel}}{(\omega_n)_{aluminum}} = \left(\frac{30}{10.5} \right)^{\frac{1}{2}} = 1.6903 = \frac{1}{0.59161}$$

Thus the natural frequency is reduced to 59.161% of its value if aluminum is used instead of steel.

2.64 For free vibration, apply Newton's second law of motion:



$$m l \ddot{\theta} + m g \sin\theta = 0 \qquad \text{(E.1)}$$

For small angular displacements, Eq. (E.1) reduces to

$$m l \ddot{\theta} + m g \theta = 0 \qquad \text{(E.2)}$$

or

$$\ddot{\theta} + \omega_n^2 \theta = 0 \qquad \text{(E.3)}$$

where

$$\omega_n = \sqrt{\frac{g}{l}} \qquad \text{(E.4)}$$

Solution of Eq. (E.3) is:

$$\theta(t) = \theta_0 \cos\omega_n t + \frac{\dot{\theta}_0}{\omega_n} \sin\omega_n t \qquad \text{(E.5)}$$

where θ_0 and $\dot{\theta}_0$ denote the angular displacement and angular velocity at $t = 0$. The amplitude of motion is given by

$$\Theta = \left\{ \theta_0^2 + \left(\frac{\dot{\theta}_0}{\omega_n} \right)^2 \right\}^{\frac{1}{2}} \qquad \text{(E.6)}$$

Using $\Theta = 0.5$ rad, $\theta_0 = 0$ and $\dot{\theta}_0 = 1$ rad/s, Eq. (E.6) gives

$$0.5 = \frac{\dot{\theta}_0}{\omega_n} = \frac{1}{\omega_n} \quad \text{or} \quad \omega_n = 2 \text{ rad/s}$$

2.65 The system of Fig. (A) can be drawn in equivalent form as shown in Fig. (B) where both pulleys have the same radius r_1. We notice in Fig. (B) that vibration can take place in only one way with one pulley moving clockwise and the other moving counter clockwise.†

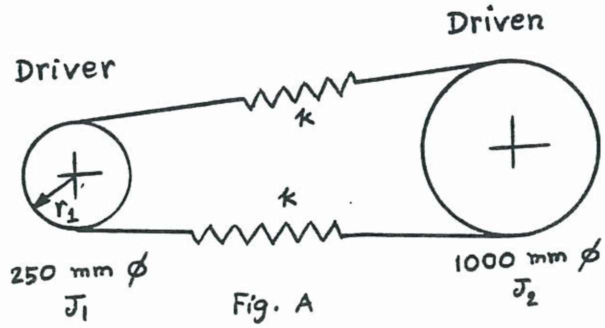


Fig. A

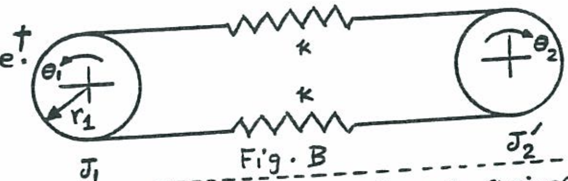


Fig. B

When pulleys rotate in opposite directions, $\dfrac{\theta_1}{\theta_2} = \dfrac{J_2}{J_1}$.

The spring force, which has the same value on either pulley is $-k_t(\theta_1+\theta_2)$ where k_t = torsional spring constant of the system. Equation of motion is

$$J_1\ddot{\theta}_1 + k_t(\theta_1+\theta_2) = 0 \quad \& \quad J_2\ddot{\theta}_2 + k_t(\theta_1+\theta_2) = 0$$

i.e. $J_1\ddot{\theta}_1 + k_t\left(1+\frac{J_1}{J_2}\right)\theta_1 = 0 \quad \& \quad J_2\ddot{\theta}_2 + k_t\left(\frac{J_2}{J_1}+1\right)\theta_2 = 0$

$$k_t = \frac{\Delta M_t}{\Delta\theta} = \left(\text{force in springs due to } \Delta\theta\right)\frac{r_1}{\Delta\theta}$$

$$= (2k\, r_1 \Delta\theta)\frac{r_1}{\Delta\theta} = 2k r_1^2$$

$$= 2k\left(\frac{125}{1000}\right)^2 = k/32 \text{ N-m/rad}$$

$\therefore$ Eq. (E_1) gives, for $\omega = 12\pi$ rad,

$k = 454.7935$ N/m.

Either of these equations gives

$$\omega = \left\{k_t\left(\frac{J_1+J_2}{J_1 J_2}\right)\right\}^{1/2} \quad --- (E_1)$$

Here $J_1 = 0.2/4 = 0.05$ kg-m^2,

$J_2' = J_2(\text{speed ratio})^2 = 0.2\left(\frac{1}{4}\right)^2 = 0.0125$ kg-m^2

† The other possible motion is rotation of the two pulleys as a whole (as rigid body) in same direction. This will have a natural frequency of zero. See section 5.7.

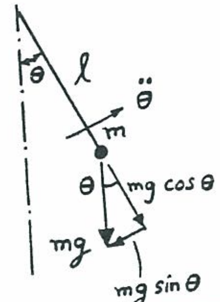
2.66

$m l \ddot{\theta} + mg \sin \theta = 0$

For small θ, $\quad m l \ddot{\theta} + mg \theta = 0$

$$\omega_n = \sqrt{\frac{g}{l}}$$

$$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{\frac{9.81}{0.5}}} = 1.4185 \text{ sec}$$



2.67

(a) $\omega_n = \sqrt{\frac{g}{l}}$

(b) $m l^2 \ddot{\theta} + k a^2 \sin \theta + mgl \sin\theta = 0 ; \quad m l^2 \ddot{\theta} + (k a^2 + mgl)\, \theta = 0$

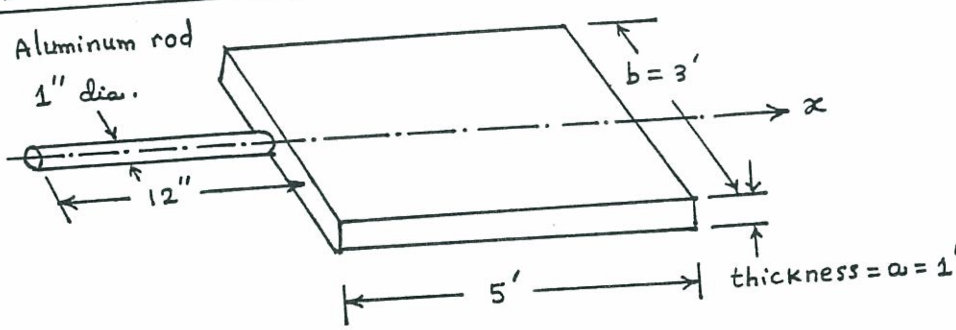
$$\omega_n = \sqrt{\frac{k a^2 + mgl}{m l^2}}$$

(c) $m l^2 \ddot{\theta} + k a^2 \sin\theta - mgl \sin \theta = 0 ; \quad m l^2 \ddot{\theta} + (k a^2 - mgl)\, \theta = 0$

$$\omega_n = \sqrt{\frac{k a^2 - mgl}{m l^2}}$$

configuration (b) has the highest natural frequency.

2.68



$$m = \text{mass of a panel} = (5 \times 12)\,(3 \times 12)\,(1)\,\left(\frac{0.283}{386.4}\right) = 1.5820$$

$$J_0 = \text{mass moment of inertia of panel about x-axis} = \frac{m}{12}\,(a^2 + b^2)$$

$$= \frac{1.5820}{12}\,(1^2 + 36^2) = 170.9878$$

$$I_0 = \text{polar moment of inertia of rod} = \frac{\pi}{32}\, d^4 = \frac{\pi}{32}\,(1)^4 = 0.098175 \text{ in}^4$$

$$k_t = \frac{G\,I_0}{\ell} = \frac{(3.8\,(10^6))\,(0.098175)}{12} = 3.1089\,(10^4)\text{ lb–in/rad}$$

$$\omega_n = \left\{\frac{k_t}{J_0}\right\}^{\frac{1}{2}} = \left\{\frac{3.1089\,(10^4)}{170.9878}\right\}^{\frac{1}{2}} = 13.4841\text{ rad/sec}$$

2.69

I_0 = polar moment of inertia of cross section of shaft AB

$$= \frac{\pi}{32}\,d^4 = \frac{\pi}{32}\,(1)^4 = 0.098175\text{ in}^4$$

k_t = torsional stiffness of shaft AB $= \dfrac{G\,I_0}{\ell}$

$$= \frac{(12\,(10^6))\,(0.098175)}{6} = 19.635\,(10^4)\text{ lb–in/rad}$$

J_0 = mass moment of inertia of the three blades about y–axis

$$= 3\,J_0\,|_{PQ} = 3\left(\frac{1}{3}\,m\,\ell^2\right) = m\,\ell^2 = \left(\frac{2}{386.4}\right)(12)^2 = 0.7453$$

Torsional natural frequency:

$$\omega_n = \left\{\frac{k_t}{J_0}\right\}^{\frac{1}{2}} = \left\{\frac{19.635\,(10^4)}{0.7453}\right\}^{\frac{1}{2}} = 513.2747\text{ rad/sec}$$

2.70

J_0 = mass moment of inertia of the ring = 1.0 kg–m².

I_{os} = polar moment of inertia of the cross section of steel shaft

$$= \frac{\pi}{32}\,(d_{os}^4 - d_{is}^4) = \frac{\pi}{4}\,(0.05^4 - 0.04^4) = 36.2266\,(10^{-8})\text{ m}^4$$

I_{ob} = polar moment of inertia of cross section of brass shaft

$$= \frac{\pi}{32}\,(d_{ob}^4 - d_{ib}^4) = \frac{\pi}{32}\,(0.04^4 - 0.03^4) = 17.1806\,(10^{-8})\text{ m}^4$$

k_{ts} = torsional stiffness of steel shaft

$$= \frac{G_s\,I_{os}}{\ell} = \frac{(80\,(10^9))\,(36.2266\,(10^{-8}))}{2} = 14490.64\text{ N–m/rad}$$

k_{tb} = torsional stiffness of brass shaft

$$= \frac{G_b\,I_{ob}}{\ell} = \frac{(40\,(10^9))\,(17.1806\,(10^{-8}))}{2} = 3436.12\text{ N–m/rad}$$

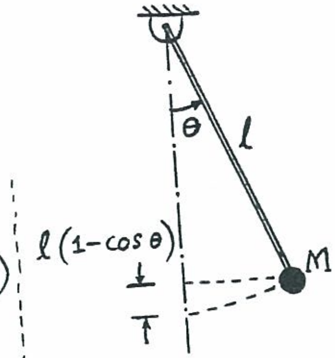
$$k_{t_{eq}} = k_{ts} + k_{tb} = 17{,}926.76\text{ N–m/rad}$$

Torsional natural frequency:

$$\omega_n = \sqrt{\frac{k_{t_{eq}}}{J_0}} = \sqrt{\frac{17926.76}{1}} = 133.8908\text{ rad/sec}$$

Natural time period:

$$\tau_n = \frac{2\,\pi}{\omega_n} = \frac{2\,\pi}{133.8908} = 0.04693\text{ sec}$$



2.71 Kinetic energy of system is

$$T = T_{rod} + T_{bob} = \frac{1}{2}\left(\frac{1}{3} m l^2\right)\dot{\theta}^2 + \frac{1}{2} M l^2 \dot{\theta}^2$$

Potential energy of system is (since mass of the rod acts through its center)

$$U = U_{rod} + U_{bob} = \frac{1}{2} m g l (1-\cos\theta) + \frac{1}{2} M g l (1-\cos\theta)$$

Equation of motion:

$$\frac{d}{dt}(T+U) = 0$$

i.e. $$\left(M + \frac{m}{3}\right) l^2 \ddot{\theta} + \left(M + \frac{m}{2}\right) g l \sin\theta = 0$$

For small angles, $$\ddot{\theta} + \frac{\left(M+\frac{m}{2}\right)}{\left(M+\frac{m}{3}\right)} \frac{g}{l} \cdot \theta = 0$$

$$\omega_n = \sqrt{\frac{\left(M+\frac{m}{2}\right) g}{\left(M+\frac{m}{3}\right) l}}$$

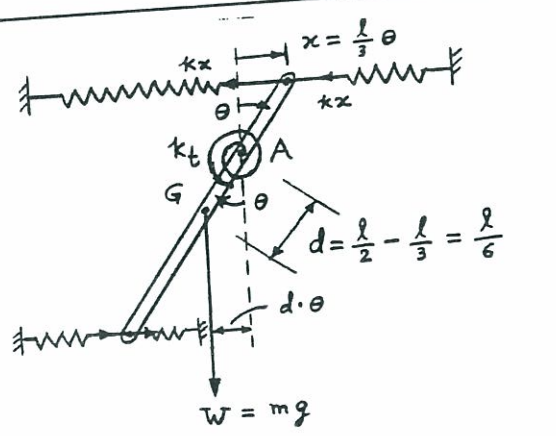
2.72 For the shaft, $$J = \frac{\pi d^4}{32} = \frac{\pi (0.05)^4}{32} = 61.3594 \times 10^{-8}\ m^4$$

$$k_t = \frac{GJ}{l} = \frac{(0.793 \times 10^{11})(61.3594 \times 10^{-8})}{2} = 24329.002\ \text{N-m/rad}$$

For the disc, $$J_0 = \frac{M D^2}{8} = \left(\rho \frac{\pi D^2}{4} h\right) \frac{D^2}{8} = \frac{\rho \pi D^4 h}{32}$$

$$= \frac{(7.83 \times 10^3)\,\pi\,(1)^4 (0.1)}{32} = 76.8710\ \text{kg-m}^2$$

$$\omega_n = \sqrt{\frac{k_t}{J_0}} = \left(\frac{24329.002}{76.8710}\right)^{1/2} = 17.7902\ \text{rad/sec}$$



2.73 Equation of motion

$$J_A \ddot{\theta} = -W d\theta - 2k\left(\frac{l}{3}\theta\right)\frac{l}{3} - 2k\left(\frac{2l}{3}\theta\right)\frac{2l}{3} - k_t \theta$$

where

$$J_A = J_G + m d^2 = \frac{1}{12} m l^2 + m \frac{l^2}{36}$$

$$= \frac{1}{9} m l^2$$

$$\therefore \frac{m l^2}{9} \ddot{\theta} + \left(m g d + \frac{2k l^2}{9} + \frac{8k l^2}{9} + k_t\right)\theta = 0$$

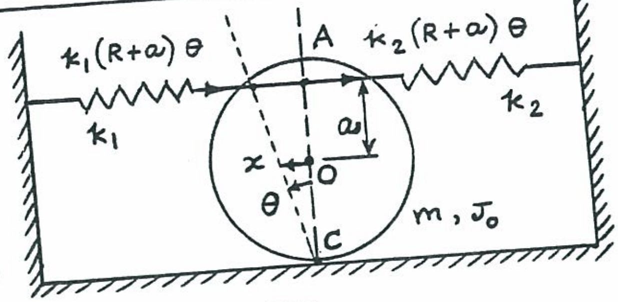
$$\omega_n = \sqrt{\frac{\left(m g d + \frac{2}{9} k l^2 + \frac{8}{9} k l^2 + k_t\right) 9}{m l^2}} = \sqrt{\frac{9 m g d + 10 k l^2 + 9 k_t}{m l^2}}$$

For given data,

$$\omega_n = \sqrt{\frac{9(10)(9.81)(5/6) + 10\,(2000)\,(5)^2 + 9\,(1000)}{10\,(5)^2}} = 45.1547 \ \frac{\text{rad}}{\text{sec}}$$

2.74

$J_0 = \frac{1}{2} m R^2$, $\quad J_C = \frac{1}{2} m R^2 + m R^2$



Let angular displacement = θ

Equation of motion:

$$J_C\,\ddot{\theta} + k_1 (R+a)^2 \theta + k_2 (R+a)^2 \theta = 0$$

$$\omega_n = \sqrt{\frac{(k_1+k_2)\,(R+a)^2}{J_C}} = \sqrt{\frac{(k_1+k_2)(R+a)^2}{1.5\ m\,R^2}} \qquad (E_1)$$

Equation (E_1) shows that ω_n increases with the value of a.

$\therefore$ ω_n will be maximum when $a = R$.

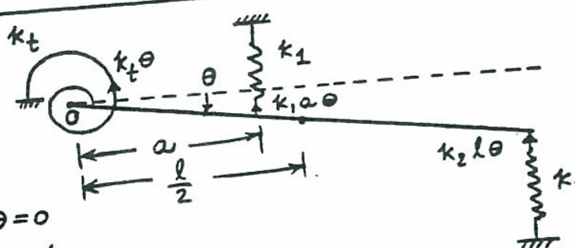
2.75

Net g acting on the pendulum $= 9.81 - 5 = 4.81 \ \text{m/sec}^2 = g_n$

$$\omega_n = \sqrt{\frac{g_n}{l}} = \sqrt{\frac{4.81}{5}} = 3.1016 \ \text{rad/sec}$$

$$\tau_n = 2\pi/\omega_n = 2.0258 \ \text{sec}$$

2.76



Equation of motion:

$$J_0\,\ddot{\theta} = -k_t\theta - (k_1 a\theta)a - (k_2 l\theta)\,l$$

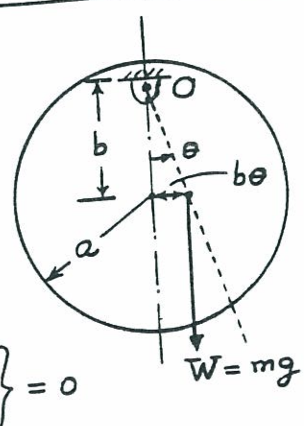
where $\quad J_0 = \frac{1}{12} m l^2 + m\left(\frac{l}{2}\right)^2 = \frac{1}{3} m l^2$

$$\therefore \quad \frac{1}{3} m l^2\,\ddot{\theta} + (k_t + k_1 a^2 + k_2 l^2)\,\theta = 0$$

$$\omega_n = \left\{\frac{3\,(k_t + k_1 a^2 + k_2 l^2)}{m l^2}\right\}^{1/2}$$

2.77

$J_0 = J_G + m b^2 = \frac{1}{2} m a^2 + m b^2$



Equation of motion:

$$J_0\,\ddot{\theta} + m g b\,\theta = 0$$

$$\omega_n = \sqrt{\frac{m g b}{J_0}} = \sqrt{\frac{2 g b}{a^2 + 2 b^2}}$$

$$\frac{\partial \omega_n}{\partial b} = \frac{1}{2}\left(\frac{2 g b}{a^2 + 2 b^2}\right)^{-\frac{1}{2}} \left\{\frac{(a^2+2b^2)(2g) - 2 g b\,(4b)}{(a^2+2b^2)^2}\right\} = 0$$

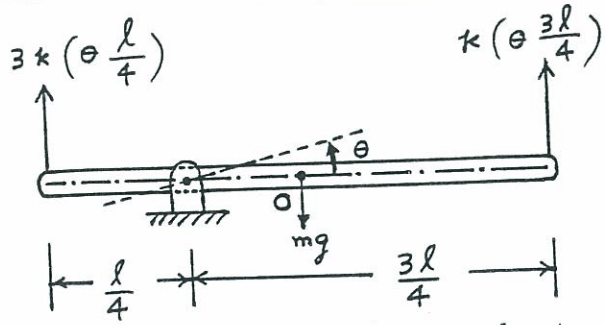
i.e., $b = \pm \dfrac{a}{\sqrt{2}}$

$$\omega_n \Big|_{b = +a/\sqrt{2}} = \sqrt{\frac{2g \frac{a}{\sqrt{2}}}{a^2 + 2(a^2/2)}} = \sqrt{\frac{g}{\sqrt{2}\, a}}$$

$b = -a/\sqrt{2}$ gives imaginary value for ω_n.

Since $\omega_n = 0$ when $b = 0$, we have $\omega_n\big|_{max}$ at $b = \dfrac{a}{\sqrt{2}}$.

2.78



Let θ be measured from static equilibrium position so that gravity force need not be considered.

(a) Newton's second law of motion:

$$J_0\, \ddot{\theta} = -3\,k\,(\theta\,\frac{\ell}{4})\,\frac{\ell}{4} - k\,(\theta\,\frac{3\,\ell}{4})\,(\frac{3\,\ell}{4}) \quad \text{or} \quad J_0\,\ddot{\theta} + \frac{3}{4}\,k\,\ell^2\,\theta = 0$$

(b) D'Alembert's principle:

$$M(t) - J_0\,\ddot{\theta} = 0 \quad \text{or} \quad -3\,k\,(\theta\,\frac{\ell}{4})\,(\frac{\ell}{4}) - k\,(\theta\,\frac{3\,\ell}{4})\,(\frac{3\,\ell}{4}) - J_0\,\ddot{\theta} = 0$$

$$\text{or} \quad J_0\,\ddot{\theta} + \frac{3}{4}\,k\,\ell^2\,\theta = 0$$

(c) Principle of virtual work:

Virtual work done by spring force:

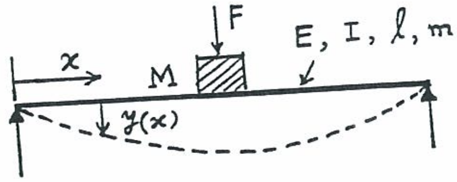
$$\delta W_s = -3\,k\,(\theta\,\frac{\ell}{4})\,(\frac{\ell}{4}\,\delta\theta) - k\,(\theta\,\frac{3\,\ell}{4})\,(\frac{3\,\ell}{4}\,\delta\theta)$$

Virtual work done by inertia moment = - $(J_0\,\ddot{\theta})\,\delta\theta$

Setting total virtual work done by all forces/moments equal to zero, we obtain

$$J_0\,\ddot{\theta} + \frac{3}{4}\,k\,\ell^2\,\theta = 0$$

2.86 Let m_{eff} = effective part of mass of beam (m) at middle. Thus vibratory inertia force at middle is due to $(M + m_{eff})$. Assume a deflection shape: $y(x,t) = Y(x) \cos(\omega_n t - \phi)$ where $Y(x)$ = static deflection shape due to load at middle given by:



$$Y(x) = Y_0 \left(3 \frac{x}{\ell} - 4 \frac{x^3}{\ell^3} \right) \; ; \; 0 \leq x \leq \frac{\ell}{2}$$

where Y_0 = maximum deflection of the beam at middle $= \dfrac{F \ell^3}{48 E I}$

Maximum strain energy of beam = maximum work done by force F $= \dfrac{1}{2} F Y_0$.

Maximum kinetic energy due to distributed mass of beam:

$$= 2 \left\{ \frac{1}{2} \frac{m}{\ell} \int_0^{\frac{\ell}{2}} \dot{y}^2(x,t) \,\Big|_{max} dx \right\} + \frac{1}{2} \left(\dot{y}_{max} \right)^2 M$$

$$= \frac{m \omega_n^2}{\ell} \int_0^{\frac{\ell}{2}} Y^2(x)\, dx + \frac{1}{2} \omega_n^2 Y_{max}^2 M$$

$$= \frac{m \omega_n^2}{\ell} \int_0^{\frac{\ell}{2}} Y_0^2 \left(\frac{9 x^2}{\ell^2} + 16 \frac{x^6}{\ell^6} - 24 \frac{x^4}{\ell^4} \right) dx + \frac{1}{2} Y_0^2 M \omega_n^2$$

$$= \frac{m \omega_n^2 Y_0^2}{\ell} \left[\frac{9}{\ell^2} \frac{x^3}{3} + \frac{16}{\ell^6} \frac{x^7}{7} - \frac{24}{\ell^4} \frac{x^5}{5} \right] \Big|_0^{\frac{\ell}{2}} + \frac{1}{2} Y_0^2 M \omega_n^2$$

$$= \frac{1}{2} Y_0^2 \omega_n^2 \left(\frac{17}{35} m + M \right)$$

This shows that $m_{eff} = \dfrac{17}{35} m = 0.4857\, m$

2.87 For small angular rotation of bar PQ about P,

$$\frac{1}{2} (k_{12})_{eq} (\theta l_3)^2 = \frac{1}{2} k_1 (\theta l_1)^2 + \frac{1}{2} k_2 (\theta l_2)^2$$

$$(k_{12})_{eq} = \frac{k_1 l_1^2 + k_2 l_2^2}{l_3^2}$$

Since $(k_{12})_{eq}$ and k_3 are in series,

$$k_{eq} = \frac{(k_{12})_{eq}\, k_3}{(k_{12})_{eq} + k_3} = \frac{k_1 k_3 l_1^2 + k_2 k_3 l_2^2}{k_1 l_1^2 + k_2 l_2^2 + k_3 l_3^2}$$

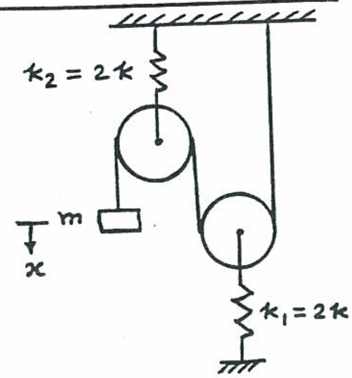
T = kinetic energy $= \frac{1}{2} m \dot{x}^2$, U = potential energy $= \frac{1}{2} k_{eq} x^2$

If $x = X \cos \omega_n t$,

$T_{max} = \frac{1}{2} m \omega_n^2 X^2$, $U_{max} = \frac{1}{2} k_{eq} X^2$

$T_{max} = U_{max}$ gives $\omega_n = \sqrt{\dfrac{k_1 k_3 l_1^2 + k_2 k_3 l_2^2}{m(k_1 l_1^2 + k_2 l_2^2 + k_3 l_3^2)}}$

2.88 When mass m moves by x, spring k_1 deflects by $x/4$.



$T =$ kinetic energy $= \frac{1}{2} m (\dot{x})^2$

$U =$ potential energy $= 2\left\{\frac{1}{2}(2k)\left(\frac{x}{4}\right)^2\right\}$

$= \frac{1}{8} k x^2$

For harmonic motion,

$T_{max} = \frac{1}{2} m \omega_n^2 X^2, \quad U_{max} = \frac{1}{8} k X^2$

$T_{max} = U_{max}$ gives $\omega_n = \sqrt{\dfrac{k}{4m}}$

2.89 Refer to the figure of solution of problem 2.24.

$T = \frac{1}{2} m \dot{x}^2, \quad U = \frac{1}{2}\left[2k_1 (x \cos 45°)^2 + 2k_2 (x \cos 135°)^2\right]$

$= \frac{1}{2}(k_1 + k_2) x^2$

For harmonic motion,

$T_{max} = \frac{1}{2} m \omega_n^2 X^2, \quad U_{max} = \frac{1}{2}(k_1 + k_2) X^2$

$T_{max} = U_{max}$ gives $\omega_n = \sqrt{\dfrac{k_1 + k_2}{m}}$

2.90 Kinetic energy (K.E.) $= \frac{1}{2} m \dot{x}^2$

Potential energy (P.E.) $= \frac{1}{2} T_1 x + \frac{1}{2} T_2 x$ = work done in displacing mass m by distance x against the total force (tension) of $T_1 + T_2$.

$T_1 = \frac{x}{a} T, \quad T_2 = \frac{x}{b} T$ from solution of problem 2.26

Max. K.E. $= \frac{1}{2} m \omega_n^2 X^2$, Max. P.E. $= \frac{1}{2} T \left(\frac{1}{a} + \frac{1}{b}\right) X^2$

Max. K.E. = Max. P.E. gives $\omega_n = \sqrt{\dfrac{T(a+b)}{m a b}} = \sqrt{\dfrac{T l}{m a (l-a)}}$

2.91 $T = K.E. = \frac{1}{2} J_A \dot{\theta}^2 = \frac{1}{2}(J_G + m d^2)\dot{\theta}^2 = \frac{1}{2}\left(\frac{1}{12} m l^2 + m \frac{l^2}{36}\right)\dot{\theta}^2$

$= \frac{1}{2}\left(\frac{m l^2}{9}\right)\dot{\theta}^2$

$U = P.E. = m g d (1 - \cos\theta) + 2\left(\frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2\right) + \frac{1}{2} k_t \theta^2$

with $\cos\theta \simeq 1 - \frac{1}{2}\theta^2$, $x_1 = \frac{l}{3}\theta$ and $x_2 = \frac{2l}{3}\theta$

$$U = mg\,\frac{l}{6}\,\frac{\theta^2}{2} + k\,\frac{l^2}{9}\,\theta^2 + k\,\frac{4l^2}{9}\,\theta^2 + \frac{1}{2}\,k_t\,\theta^2$$

$$T_{max} = \frac{1}{2}\left(\frac{ml^2}{9}\right)\Theta^2, \quad U_{max} = \frac{1}{2}\,\frac{mgl}{6}\,\Theta^2 + \frac{1}{2}\left(\frac{10\,k\,l^2}{9}\right)\Theta^2 + \frac{1}{2}k_t\,\Theta^2$$

$T_{max} = U_{max}$ gives

$$\omega_n = \sqrt{\frac{mg\frac{l}{6} + \frac{10\,k\,l^2}{9} + k_t}{\frac{ml^2}{9}}} = 45.1547\ \frac{rad}{sec} \text{ for given data}$$

2.92 Refer to the figure in the solution of problem 2.76

$$T = \frac{1}{2}\,J_0\,\dot{\theta}^2$$

$$U = \frac{1}{2}\,k_t\,\theta^2 + \frac{1}{2}\,k_1\,(\theta a)^2 + \frac{1}{2}\,k_2\,(\theta\,l)^2$$

For $\theta(t) = \Theta\cos\omega_n t$,

$$T_{max} = \frac{1}{2}\,J_0\,\omega_n^2\,\Theta^2, \quad U_{max} = \frac{1}{2}\left(k_t + k_1 a^2 + k_2 l^2\right)\Theta^2$$

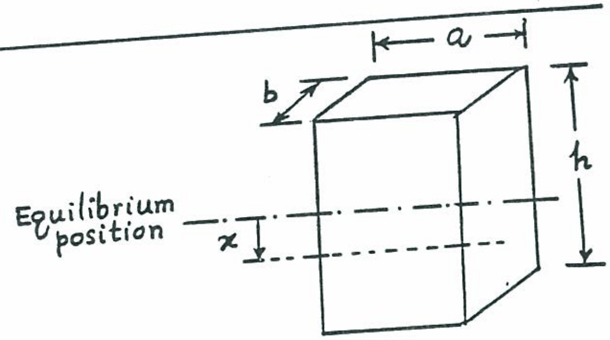
$T_{max} = U_{max}$ gives

$$\omega_n = \sqrt{\frac{k_t + k_1 a^2 + k_2 l^2}{J_0}} = \sqrt{\frac{3\,(k_t + k_1 a^2 + k_2 l^2)}{m\,l^2}}$$

since $J_0 = ml^2/3$.

2.93 When prism is displaced by x from equilibrium position, the weight of oil displaced

$= \rho_o\,g\,ab\,x$ = restoring force



Mass of prism $= m = \rho_w\,abh$

Equation of motion:

$m\ddot{x}$ + restoring force $= 0$

$$\rho_w\,abh\,\ddot{x} + \rho_o\,g\,ab\,x = 0 \qquad (E_1)$$

$$\omega_n = \sqrt{\frac{\rho_o\,g\,a\,b}{\rho_w\,a\,b\,h}} = \sqrt{\frac{\rho_o\,g}{\rho_w\,h}}$$

Since ω_n is independent of cross-section of the prism, ω_n remains same even for a circular wooden prism.

2.94

$$T = \frac{1}{2}\,m\,\dot{x}^2 + \frac{1}{2}\,J_0\,\dot{\theta}^2 = \frac{1}{2}\left(m\,R^2 + \frac{1}{2}\,m\,R^2\right)\dot{\theta}^2$$

since $x = R\,\theta$ and $J_0 = \frac{1}{2}\,m\,R^2$.

$$U = \frac{1}{2}\,k_1\,x_1^2 + \frac{1}{2}\,k_2\,x_1^2 = \frac{1}{2}\,(k_1 + k_2)\,(R + a)^2\,\theta^2$$

where $x_1 = (R + a)\,\theta$. Using $\frac{d}{dt}(T + U) = 0$, we obtain

$$\left(\frac{3}{2}\, m\, R^2\right) \ddot{\theta} + (k_1 + k_2)\,(R + a)^2\, \theta = 0$$

2.95 Let x(t) be measured from static equilibrium position of mass. T = kinetic energy of the system:

$$T = \frac{1}{2}\, m\, \dot{x}^2 + \frac{1}{2}\, J_0\, \dot{\theta}^2 = \frac{1}{2}\left(m + \frac{J_0}{r^2}\right) \dot{x}^2$$

since $\dot{\theta} = \frac{\dot{x}}{r}$ = angular velocity of pulley. U = potential energy of the system:

$$U = \frac{1}{2}\, k\, y^2 = \frac{1}{2}\, k\, (16\, x^2)$$

since $y = \theta\,(4\,r) = 4\,x$ = deflection of spring. $\frac{d}{dt}(T + U) = 0$ leads to:

$$m\,\ddot{x} + \frac{J_0}{r^2}\,\ddot{x} + 16\,k\,x = 0$$

This gives the natural frequency:

$$\omega_n = \sqrt{\frac{16\,k\,r^2}{m\,r^2 + J_0}}$$

2.97 For pendulum, $\omega_n = \sqrt{g/l}$ in vaccum $= 0.5$ Hz $= \pi$ rad/sec

$$l = g/\pi^2 = 9.81/\pi^2 = 0.9940 \text{ m}$$

$\omega_d = \omega_n \sqrt{1-\zeta^2}$ in viscous medium $= 0.45$ Hz $= 0.9\pi$ rad/sec

$$\zeta^2 = \frac{\omega_n^2 - \omega_d^2}{\omega_n^2} = \pi^2\left(\frac{1-0.81}{1}\right) = 1.8752$$

$$\zeta = 1.3694$$

Equation of motion: $\quad m l^2 \ddot{\theta} + c_t \dot{\theta} + m g l \theta = 0$

$$c_{ct} = 2(m l^2)\,\omega_n = 2(1)(0.994)^2(\pi) = 6.2080$$

Since $\zeta = \dfrac{c_t}{c_{ct}} = 1.3694$, $\quad c_t = 8.5013$ N-m-sec/rad.

2.98 From Eq. (2.85),

$$\ln\left(\frac{x_j}{x_{j+1}}\right) = \ln(18) \Rightarrow \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = 2.8904$$

$$\zeta = \left\{\frac{(2.8904)^2}{(2.8904)^2 + 4\pi^2}\right\}^{\frac{1}{2}} = 0.4179$$

(a) If damping is doubled, $\zeta_{new} = 0.8358$

$$\ln\left(\frac{x_j}{x_{j+1}}\right) = \frac{2\pi\,\zeta_{new}}{\sqrt{1-\zeta_{new}^2}} = \frac{2\pi(0.8358)}{\sqrt{1-(0.8358)^2}} = 9.5656$$

$$\therefore \frac{x_j}{x_{j+1}} = 14265.362$$

(b) If damping is halved, $\zeta = 0.2090$

$$\ln\left(\frac{x_j}{x_{j+1}}\right) = \frac{2\pi\,\zeta_{new}}{\sqrt{1-\zeta_{new}^2}} = \frac{2\pi(0.2090)}{\sqrt{1-(0.2090)^2}} = 1.3428$$

$$\therefore \frac{x_j}{x_{j+1}} = 3.8296$$

2.99

$$x(t) = X\, e^{-\zeta\omega_n t} \sin\omega_d t \quad \text{where} \quad \omega_d = \sqrt{1-\zeta^2}\,\omega_n$$

For maximum or minimum of $x(t)$,

$$\frac{dx}{dt} = X\, e^{-\zeta\omega_n t}\left(-\zeta\omega_n \sin\omega_d t + \omega_d \cos\omega_d t\right) = 0$$

As $e^{-\zeta\omega_n t} \neq 0$ for finite t,

$$-\zeta\omega_n \sin\omega_d t + \omega_d \cos\omega_d t = 0$$

i.e. $\tan\omega_d t = \sqrt{1-\zeta^2}/\zeta$

Using the relation

$$\sin\omega_d t = \pm\frac{\tan\omega_d t}{\sqrt{1+\tan^2\omega_d t}} = \pm\frac{\left(\sqrt{1-\zeta^2}/\zeta\right)}{\sqrt{1+\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)^2}} = \pm\sqrt{1-\zeta^2}$$

we obtain

$$\sin\omega_d t = \sqrt{1-\zeta^2}, \quad \cos\omega_d t = \zeta$$

and

$$\sin\omega_d t = -\sqrt{1-\zeta^2}, \quad \cos\omega_d t = -\zeta$$

$$\frac{d^2x}{dt^2} = X\, e^{-\zeta\omega_n t}\left[\zeta^2\omega_n^2 \sin\omega_d t - 2\zeta\omega_n\omega_d \cos\omega_d t - \omega_d^2 \sin\omega_d t\right]$$

When $\sin\omega_d t = \sqrt{1-\zeta^2}$ and $\cos\omega_d t = \zeta$,

$$\frac{d^2x}{dt^2} = -X\, e^{-\zeta\omega_n t}\,\omega_n^2\sqrt{1-\zeta^2} < 0$$

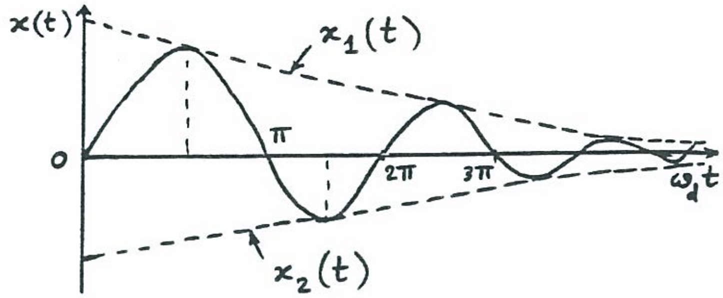
$\therefore \sin\omega_d t = \sqrt{1-\zeta^2}$ corresponds to maximum of $x(t)$.

When $\sin\omega_d t = -\sqrt{1-\zeta^2}$ and $\cos\omega_d t = -\zeta$,

$$\frac{d^2x}{dt^2} = X\, e^{-\zeta \omega_n t}\, \omega_n^2 \sqrt{1-\zeta^2} > 0$$

$\therefore \sin \omega_d t = -\sqrt{1-\zeta^2}$ corresponds to minimum of $x(t)$.

Enveloping curves:



Let the curve passing through the maximum (or minimum) points be

$$x(t) = C\, e^{-\zeta \omega_n t}$$

For maximum points, $t_{max} = \dfrac{\sin^{-1}\left(\sqrt{1-\zeta^2}\right)}{\omega_d}$

and

$$C\, e^{-\zeta \omega_n t_{max}} = X\, e^{-\zeta \omega_n t_{max}} \sin \omega_d t_{max}$$

i.e.

$$C = X \sqrt{1-\zeta^2}$$

$$\therefore x_1(t) = X \sqrt{1-\zeta^2}\, e^{-\zeta \omega_n t}$$

Similarly for minimum points, $t_{min} = \dfrac{\sin^{-1}\left(-\sqrt{1-\zeta^2}\right)}{\omega_d}$

and

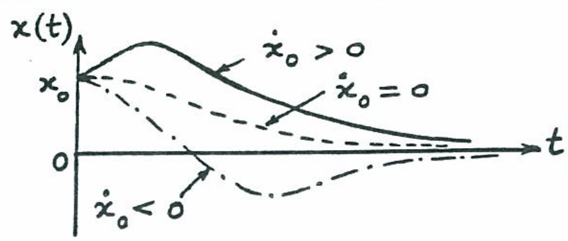
$$C\, e^{-\zeta \omega_n t_{min}} = X\, e^{-\zeta \omega_n t_{min}} \sin \omega_d t_{min}$$

i.e.

$$C = -X\sqrt{1-\zeta^2}$$

$$\therefore x_2(t) = -X\sqrt{1-\zeta^2}\, e^{-\zeta \omega_n t}$$

2.100 $x(t) = \left[x_0 + (\dot{x}_0 + \omega_n x_0)t\right] e^{-\omega_n t}$ ---- (E_1)



For $x_0 > 0$, graph of Eq. (E_1) is shown for different $\dot{x}_0$.

We assume $\dot{x}_0 > 0$ as it is the only case that gives a maximum.

For maximum of $x(t)$,

$$\frac{dx}{dt} = e^{-\omega_n t}\left\{-(\dot{x}_0 + \omega_n x_0)\,\omega_n t + \dot{x}_0\right\} = 0$$

$$t_m = \frac{\dot{x}_0}{\omega_n(\dot{x}_0 + \omega_n x_0)} \quad \text{---- } (E_2)$$

$$\frac{d^2x}{dt^2} = -e^{-\omega_n t}\left\{2\omega_n \dot{x}_0 + \omega_n^2 x_0 - \omega_n^2(\dot{x}_0 + \omega_n x_0)t\right\} \quad ---- (E_3)$$

(E_2) and (E_3) give

$$\left.\frac{d^2x}{dt^2}\right|_{t=t_m} = -e^{-\omega_n t_m}\left\{2\omega_n \dot{x}_0 + \omega_n^2 x_0 - \omega_n^2(\dot{x}_0 + \omega_n x_0)t_m\right\}$$

$$= -e^{-\omega_n\left(\frac{\dot{x}_0}{\omega_n(\dot{x}_0 + \omega_n x_0)}\right)}\left\{\omega_n \dot{x}_0 + \omega_n^2 x_0\right\} \quad ---- (E_4)$$

For $x_0 > 0$ and $\dot{x}_0 > 0$, $\left.\frac{d^2x}{dt^2}\right|_{t_m} < 0$

Hence t_m given by Eq. (E_2) corresponds to a maximum of $x(t)$.

$$x\big|_{t=t_m} = \left\{x_0 + (\dot{x}_0 + \omega_n x_0)\frac{\dot{x}_0}{\omega_n(\dot{x}_0 + \omega_n x_0)}\right\} e^{-\omega_n t_m}$$

$$= \left(x_0 + \frac{\dot{x}_0}{\omega_n}\right) e^{-\left(\frac{\dot{x}_0}{\dot{x}_0 + \omega_n x_0}\right)} \quad ---- (E_5)$$

2.101 Equation (2.92) can be expressed as $\delta = \frac{1}{m}\ln\left(\frac{x_0}{x_m}\right)$

For half cycle, $m = \frac{1}{2}$ and hence $\delta = 2\ln\left(\frac{x_0}{x_{\frac{1}{2}}}\right) = 2\ln\left(\frac{1}{0.15}\right)$

$= 3.7942$

Necessary damping ratio ζ_0 is

$$\zeta_0 = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} = \frac{3.7942^2}{\sqrt{4\pi^2 + 3.7942^2}}$$

$$= 0.5169$$

(a)
If $\zeta = \frac{3}{4}\zeta_0 = 0.3877$, the overshoot can be determined by finding δ from Eq. (2.85):

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi\,(0.3877)}{\sqrt{1 - 0.3877^2}} = 2.6427 = 2\ln\left(\frac{x_0}{x_{\frac{1}{2}}}\right)$$

$$\ln\left(\frac{x_0}{x_{\frac{1}{2}}}\right) = 1.32135$$

$$x_{\frac{1}{2}} = x_0 / e^{1.32135} = 0.266775\ x_0$$

$\therefore$ overshoot is 26.6775%

(b)
If $\zeta = \frac{5}{4}\zeta_0 = 0.6461$, δ is given by

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi\,(0.6461)}{\sqrt{1 - (0.6461)^2}} = 5.3189 = 2\ln\left(\frac{x_0}{x_{\frac{1}{2}}}\right)$$

$$\frac{x_0}{x_{\frac{1}{2}}} = 14.2888 \quad , \quad x_{\frac{1}{2}} = 0.0700 \; x_0$$

$\therefore$ overshoot = 7%

2.102

(i) (a) Viscous damping, (b) Coulomb damping.

(iii) (a) $\tau_d = 0.2$ sec, $f_d = 5$ Hz, $\omega_d = 31.416$ rad/sec.

(b) $\tau_n = 0.2$ sec, $f_n = 5$ Hz, $\omega_n = 31.416$ rad/sec.

(ii) (a) $\dfrac{x_i}{x_{i+1}} = e^{\zeta \omega_n \tau_d}$

$$\ln\left(\frac{x_i}{x_{i+1}}\right) = \ln 2 = 0.6931 = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

or $39.9590\,\zeta^2 = 0.4804$ or $\zeta = 0.1096$

Since $\omega_d = \omega_n \sqrt{1-\zeta^2}$, we find

$$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = \frac{31.416}{\sqrt{0.98798}} = 31.6065 \text{ rad/sec}$$

$$k = m\,\omega_n^2 = \left(\frac{500}{9.81}\right)(31.6065)^2 = 5.0916\,(10^4) \text{ N/m}$$

$$\zeta = \frac{c}{c_c} = \frac{c}{2\,m\,\omega_n}$$

Hence $c = 2\,m\,\omega_n\,\zeta = 2\left(\dfrac{500}{9.81}\right)(31.6065)(0.1096) = 353.1164$ N–s/m

(b) From Eq. (2.135):

$$k = m\,\omega_n^2 = \frac{500}{9.81}(31.416)^2 = 5.0304\,(10^4) \text{ N/m}$$

Using N = W = 500 N,

$$\mu = \frac{0.002\,k}{4\,W} = \frac{(0.002)\,(5.0304\,(10^4))}{4\,(500)} = 0.0503$$

2.103

(a) $c_c = 2\sqrt{km} = 2\sqrt{5000 \times 50} = 1000$ N-s/m

(b) $c = c_c/2 = 500$ N-s/m

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \sqrt{\frac{k}{m}}\sqrt{1-\left(\frac{c}{c_c}\right)^2} = \sqrt{\frac{5000}{50}}\sqrt{1-\left(\frac{1}{2}\right)^2}$$

$= 8.6603$ rad/sec

(c) From Eq. (2.85), $\delta = \dfrac{2\pi}{\omega_d}\left(\dfrac{c}{2m}\right) = \dfrac{2\pi}{8.6603}\left(\dfrac{500}{2\times 50}\right)$

$= 3.6276$

2.104

$m = 2000$ kg, $v = \dot{x}_0 = 10$ m/sec, $k = 40{,}000$ N/m

$c = 20{,}000$ N-sec/m

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{40000}{2000}} = 4.4721 \text{ rad/sec}$$

$$c_c = 2\sqrt{km} = 25{,}298.221 \text{ N-sec/m}$$

$$\zeta = c/c_c = 0.7906$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 4.4721\sqrt{1-(0.7906)^2} = 2.7384 \text{ rad/sec}$$

$$\tau_d = 2\pi/\omega_d = 2.2945 \text{ sec}$$

(a) For $x_0 = 0$ and $\dot{x}_0 = 10$ m/sec, Eq. (2.72) gives

$$x(t) = e^{-\zeta\omega_n t}\,\frac{\dot{x}_0}{\omega_n\sqrt{1-\zeta^2}}\,\sin\omega_n\sqrt{1-\zeta^2}\,t$$

At x_{max}, $\omega_n t \simeq \frac{\pi}{2}$ and $\sin\omega_n\sqrt{1-\zeta^2}\,t \simeq 1$

$$\therefore\ x_{max} \simeq e^{-0.7906(\pi/2)}\cdot\left(\frac{10}{2.7384}\right)\cdot(1) = 1.0548 \text{ m}$$

(b) $t = \tau_d/4 = 2.2945/4 = 0.5736$ sec.

2.105

$\omega_n = 200$ cycles/min $= 20.944$ rad/sec , $\omega_d = 180$ cycles/min $= 18.8496\ \frac{\text{rad}}{\text{sec}}$

$J_0 = 0.2$ kg-m^2

Since $\omega_d = \sqrt{1-\zeta^2}\,\omega_n$, $\zeta = \sqrt{1-\left(\frac{\omega_d}{\omega_n}\right)^2} = \sqrt{1-\left(\frac{18.8496}{20.944}\right)^2} = 0.4359$

$$= \frac{c_t}{(c_t)_{cri}} = \frac{c_t}{2J_0\omega_n}$$

$$c_t = 2J_0\omega_n\zeta = 2(0.2)(20.944)(0.4359)$$

$$= 3.6518 \text{ N-m-s/rad}$$

Eq. (2.72) can be used to obtain $\theta(t)$ for $\dot{\theta}_0 = 0$, $\theta_0 = 2° = 0.03491$ rad and $t = \tau_d = \frac{2\pi}{\omega_d} = 0.3333$ sec,

$$\theta(t) = e^{-\zeta\omega_n t}\,\theta_0\left\{\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right\}$$

$$= e^{-(0.4359)(20.944)(0.3333)}(0.03491)\left\{\cos 18.8496\times 0.3333 + \frac{0.4359\times 20.944}{18.8496}\sin 18.8496\times 0.3333\right\}$$

$$= 0.001665 \text{ rad} = 0.09541°$$

2.106

Assume that the bicycle and the boy fall as a rigid body by 5 cm at point A. Thus the mass (m_{eq}) will be subjected to an initial downward displacement of 5 cm ($t = 0$ assumed at point A):

$$x_0 = 0.05 \text{ m},\ \dot{x}_0 = 0$$

$$\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}} = \sqrt{\frac{(50000)\,(9.81)}{800}} = 24.7614 \text{ rad/sec}$$

$$c_c = 2\, m\, \omega_n = 2 \left(\frac{800}{9.81} \right) (24.7614) = 4038.5566 \text{ N—s/m}$$

$$\varsigma = \frac{c}{c_c} = \frac{1000.0}{4038.5566} = 0.2476 \text{ (underdamping)}$$

$$\omega_d = \omega_n \sqrt{1 - \varsigma^2} = 24.7614 \sqrt{1 - 0.2476^2} = 23.9905 \text{ rad/sec}$$

Response of the system:

$$x(t) = X\, e^{-\varsigma \omega_n t} \sin (\omega_d\, t + \phi)$$

where $X = \left\{ x_0^2 + \left(\frac{\dot{x}_0 + \varsigma\, \omega_n\, x_0}{\omega_d} \right)^2 \right\}^{\frac{1}{2}}$

$$= \left\{ (0.05)^2 + \left(\frac{(0.2476)\,(24.7614)\,(0.05)}{23.9905} \right)^2 \right\}^{\frac{1}{2}} = 0.051607 \text{ m}$$

and $\phi = \tan^{-1} \left(\frac{x_0\, \omega_d}{\dot{x}_0 + \varsigma\, \omega_n\, x_0} \right) = \tan^{-1} \left(\frac{0.05\ (23.9905)}{0.2476\ (24.7614)\ (0.05)} \right) = 75.6645°$

Thus the displacement of the boy (positive downward) in vertical direction is given by

$$x(t) = 0.051607\, e^{-6.1309\, t} \sin (23.9905\, t + 75.6645°) \text{ m}$$

2.107 Reduction in amplitude of viscously damped free vibration in one cycle = 0.5 in.

$$\frac{x_1}{x_2} = \frac{6.0}{5.5} = 1.0909; \quad \ln \frac{x_1}{x_2} = 0.08701 = \frac{2\, \pi\, \varsigma}{\sqrt{1 - \varsigma^2}}$$

i.e., $0.007571\, (1 - \varsigma^2) = 39.478602\, \varsigma^2$ or $\varsigma = 0.013847$

2.108 $\tau_d = 0.2$ sec $= \frac{2\pi}{\omega_d}$, $\omega_d = 31.416$ rad/sec

From Eq. (2.92) $\delta = \frac{1}{50} \ln 10 = 0.04605$

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} = \frac{0.04605}{\sqrt{(2\pi)^2 + 0.04605^2}} = 0.007329$$

When damping is neglected,

$\omega_n = \omega_d / \sqrt{1 - \zeta^2} = 31.417$ rad/sec ; $\tau_n = \frac{2\pi}{\omega_n} = 0.19999$ sec

Proportional decrease in period $= \left(\frac{0.2 - 0.19999}{0.2} \right) = 0.00005$

2.109 For critically damped system, Eq. (2.80) gives

$$x(t) = \{ x_0 + (\dot{x}_0 + \omega_n\, x_0)\, t \}\, e^{-\omega_n t} \tag{E$_1$}$$

$$\dot{x}(t) = e^{-\omega_n t} \{ \dot{x}_0 - \dot{x}_0\, \omega_n t - \omega_n^2\, x_0 t \} \tag{E$_2$}$$

Let t_m = time at which $x = x_{max}$ and $\dot{x} = 0$ occur.
Here $x_0 = 0$ and $\dot{x}_0$ = initial recoil velocity. By setting $\dot{x}(t) = 0$, Eq. (E_2) gives

$$t_m = \frac{\dot{x}_0}{\omega_n(\dot{x}_0 + \omega_n x_0)} = \frac{\dot{x}_0}{\omega_n \dot{x}_0} = \frac{1}{\omega_n} \qquad (E_3)$$

With Eq. (E_3) for t_m and $x_0 = 0$, (E_1) gives

$$x_{max} = \dot{x}_0 t_m e^{-\omega_n t_m} = \frac{\dot{x}_0 e^{-1}}{\omega_n}$$

i.e. $$\dot{x}_0 = \omega_n x_{max} e = \omega_n (0.5)(2.7183) \qquad (E_4)$$

Using $\dot{x}_0 = 10$ m/sec, $\omega_n = 10/(0.5 * 2.7183) = 7.3575 \frac{rad}{sec}$

When mass of gun is 500 kg,
the stiffness of the spring is

$$k = \omega_n^2 m = (7.3575)^2 (500) = 27{,}066.403 \text{ N/m}$$

2.110 $k = 5000$ N/m, $c_c = 0.2$ N-s/mm $= 200$ N-s/m

$$= 2\sqrt{km} = 2\sqrt{5000\ m}$$

$$m = 2 \text{ kg}$$

$$\omega_n = \sqrt{k/m} = \sqrt{5000/2} = 50 \text{ rad/sec}$$

Logarithmic decrement $= \delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = 2.0$

i.e., $\zeta = \frac{c}{c_c} = 0.3033$ and $c = 0.3033\,(0.2) = 60.66$ N-s/m

Assuming $x_0 = 0$ and $\dot{x}_0 = 1$ m/s,

$$x(t) = e^{-\zeta\omega_n t} \frac{\dot{x}_0}{\omega_n\sqrt{1-\zeta^2}} \sin\sqrt{1-\zeta^2}\,\omega_n t$$

For x_{max}, $\omega_n t \approx \pi/2$ and $\sin\sqrt{1-\zeta^2}\,\omega_n t \approx 1$

$$\therefore \quad x_{max} \simeq e^{-0.3033(\pi/2)} \frac{1}{50\sqrt{1-0.3033^2}}(1) = 0.01303 \text{ m}$$

2.111 For an overdamped system, Eq. (2.81) gives

$$x(t) = e^{-\zeta\omega_n t}(C_1 e^{\omega_d t} + C_2 e^{-\omega_d t}) \qquad (E_1)$$

Using the relations $$e^{\pm x} = \cosh x \pm \sinh x \qquad (E_2)$$

Eq. (E_1) can be rewritten as

$$x(t) = e^{-\zeta\omega_n t}(C_3 \cosh \omega_d t + C_4 \sinh \omega_d t) \qquad (E_3)$$

where $C_3 = C_1 + C_2$ and $C_4 = C_1 - C_2$.

Differentiating (E_3),

$$\dot{x}(t) = e^{-\zeta\omega_n t}\left[C_3\, \omega_d \sinh \omega_d t + C_4\, \omega_d \cosh \omega_d t\right]$$

$$- \zeta\omega_n\, e^{-\zeta\omega_n t}\left[C_3 \cosh \omega_d t + C_4 \sinh \omega_d t\right] \qquad (E_4)$$

Initial conditions $x(t=0) = x_0$ and $\dot{x}(t=0) = \dot{x}_0$ with (E_3) and (E_4) give

$$C_3 = x_0 \,, \quad C_4 = (\dot{x}_0 + \zeta\omega_n x_0)/\omega_d \qquad (E_5)$$

Thus (E_3) becomes

$$x(t) = x_0\, e^{-\zeta\omega_n t}\left(\cosh \omega_d t + \frac{\zeta\omega_n}{\omega_d} \sinh \omega_d t\right)$$

$$+ \frac{\dot{x}_0}{\omega_d}\, e^{-\zeta\omega_n t} \sinh \omega_d t \qquad (E_6)$$

(i) When $\dot{x}_0 = 0$, Eq. (E_6) gives

$$x(t) = x_0\, e^{-\zeta\omega_n t}\left(\cosh \omega_d t + \frac{\zeta\omega_n}{\omega_d} \sinh \omega_d t\right) \qquad (E_7)$$

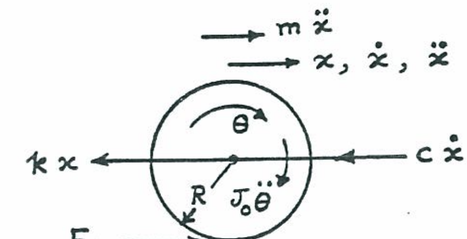
since $e^{-\zeta\omega_n t}$, $\cosh \omega_d t$, $\frac{\zeta\omega_n}{\omega_d}$ and $\sinh \omega_d t$ do not change sign (always positive) and $e^{-\zeta\omega_n t}$ approaches zero with increasing t, $x(t)$ will not change sign.

(ii) When $x_0 = 0$, Eq. (E_6) gives

$$x(t) = \frac{\dot{x}_0}{\omega_d}\, e^{-\zeta\omega_n t} \sinh \omega_d t \qquad (E_8)$$

Here also, ω_d, $e^{-\zeta\omega_n t}$ and $\sinh \omega_d t$ do not change sign (always positive) and $e^{-\zeta\omega_n t}$ approaches zero with increasing t, $x(t)$ will not change sign.

2.112 Newton's second law of motion:



$$\sum F = m\ddot{x} = -k x - c\dot{x} + F_f \qquad (1)$$

$$\sum M = J_0\, \ddot{\theta} = -F_f\, R \qquad (2)$$

where F_f = friction force.

Using $J_0 = \frac{m R^2}{2}$ and $\ddot{\theta} = \frac{\ddot{x}}{R}$, Eq. (2) gives

$$F_f = -\frac{1}{2R}\left(m R^2\right)\frac{\ddot{x}}{R} = -\frac{1}{2}\, m\ddot{x} \qquad (3)$$

Substitution of Eq. (3) into (1) yields:

$$\frac{3}{2}\,m\,\ddot{x} + c\,\dot{x} + k\,x = 0 \tag{4}$$

The undamped natural frequency is: $\omega_n = \sqrt{\frac{2\,k}{3\,m}}$ (5)

2.113 Newton's second law of motion: (measuring x from static equilibrium position of cylinder)

$$\sum F = m\,\ddot{x} = -\,k\,x - c\,\dot{x} - k\,x + F_f \tag{1}$$

$$\sum M = J_0\,\ddot{\theta} = -\,F_f\,R \tag{2}$$

where F_f = friction force. Using $J_0 = \frac{1}{2}\,m\,R^2$ and $\ddot{\theta} = \frac{\ddot{x}}{R}$, Eq. (2) gives

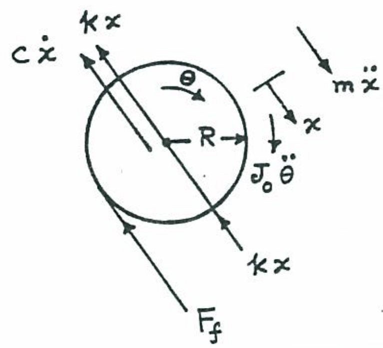
$$F_f = -\,\frac{1}{2}\,m\,\ddot{x} \tag{3}$$

Substitution of Eq. (3) into (1) gives

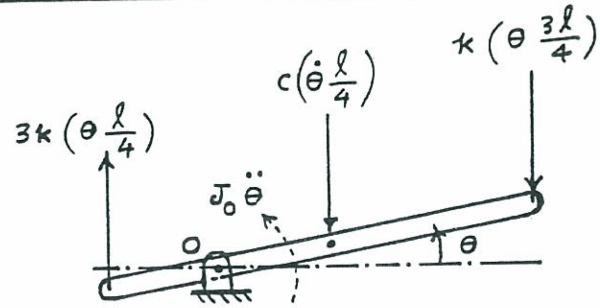
$$\frac{3}{2}\,m\,\ddot{x} + c\,\dot{x} + 2\,k\,x = 0 \tag{4}$$

Undamped natural frequency of the system:

$$\omega_n = \sqrt{\frac{4\,k}{3\,m}} \tag{4}$$



2.114



Consider a small angular displacement of the bar θ about its static equilibrium position. Newton's second law gives:

$$\sum M = J_0\,\ddot{\theta} = -k\left(\theta\,\frac{3\,\ell}{4}\right)\left(\frac{3\,\ell}{4}\right) - c\left(\dot{\theta}\,\frac{\ell}{4}\right)\left(\frac{\ell}{4}\right) - 3\,k\left(\theta\,\frac{\ell}{4}\right)\left(\frac{\ell}{4}\right)$$

$$\text{i.e.,}\quad J_0\,\ddot{\theta} + \frac{c\,\ell^2}{16}\,\dot{\theta} + \frac{3}{4}\,k\,\ell^2\,\theta = 0$$

where $J_0 = \frac{7}{48} m \ell^2$. The undamped natural frequency of torsional vibration is given by:

$$\omega_n = \sqrt{\frac{3 k \ell^2}{4 J_0}} = \sqrt{\frac{36 k}{7 m}}$$

2.115 Let δx = virtual displacement given to cylinder. Virtual work done by various forces:

Inertia forces: $\delta W_i = -(J_0 \ddot{\theta})(\delta\theta) - (m \ddot{x}) \delta x = -(J_0 \ddot{\theta})(\frac{\delta x}{R}) - (m \ddot{x}) \delta x$

Spring force: $\delta W_s = -(k x) \delta x$

Damping force: $\delta W_d = -(c \dot{x}) \delta x$

By setting the sum of virtual works equal to zero, we obtain:

$$-\frac{J_0}{R}\left(\frac{\ddot{x}}{R}\right) - m \ddot{x} - k x - c \dot{x} = 0 \quad \text{or} \quad \frac{3}{2} m \ddot{x} + c \dot{x} + k x = 0$$

2.116 Let δx = virtual displlacement given to cyllinder from its static equillibrium position. Virtualll works done by various forces:

Inertia forces: $\delta W_i = -(J_0 \ddot{\theta})\, \delta\theta - (m \ddot{x})\, \delta\theta = -(J_0 \frac{\ddot{x}}{R})(\frac{\delta x}{R}) - (m \ddot{x}) \delta x$

Spring force: $\delta W_s = -(k x) \delta x - (k x) \delta x = -2 k x \delta x$

Damping force: $\delta W_d = -(c \dot{x}) \delta x$

By setting the sum of virtual works equal to zero, we find

$$-\frac{J_0}{R} \frac{\ddot{x}}{R} - m \ddot{x} - 2 k x - c \dot{x} = 0 \tag{1}$$

Using $J_0 = \frac{1}{2} m R^2$, Eq. (1) can be rewritten as

$$\frac{3}{2} m \ddot{x} + c \dot{x} + 2 k x = 0 \tag{2}$$

2.117 See figure given in the solution of Problem 2.114. Let $\delta\theta$ be virtuall angular displacement given to the bar about its static equilibrium position. Virtual works done by various forces:

Inertia force: $\delta W_i = -(J_0 \ddot{\theta})\, \delta\theta$

Spring forces:

$$\delta W_s = -(k\, \theta \frac{3 \ell}{4})(\frac{3 \ell}{4} \delta\theta) - (3 k\, \theta \frac{\ell}{4})(\frac{\ell}{4} \delta\theta) = -\left(\frac{3}{4} k \ell^2 \theta\right) \delta\theta$$

Damping force: $\delta W_d = -(c \dot{\theta} \frac{\ell}{4})(\frac{\ell}{4} \delta\theta)$

By setting the sum of virtual works equal to zero, we get the equation of motion as:

$$J_0 \ddot{\theta} + c \frac{\ell^2}{16} \dot{\theta} + \frac{3}{4} k \ell^2 \theta = 0$$

2.118 See solution of Problem 2.93. When wooden prism is given a displacement x, equation of motion becomes: $m\ddot{x}$ + restoring force = 0 where m = mass of prism = 40 kg and restoring force = weight of fluid displaced $= \rho_0\, g\, a\, b\, x = \rho_0\,(9.81)\,(0.4)\,(0.6)\, x = 2.3544\,\rho_0\, x$ where ρ_0 is the density of the fluid. Thus the equation of motion becomes:

$$40\,\ddot{x} + 2.3544\,\rho_0\, x = 0$$

$$\text{Natural frequency} = \omega_n = \sqrt{\frac{2.3544\,\rho_0}{40}}$$

$$\text{Since } \tau_n = \frac{2\pi}{\omega_n} = 0.5, \text{we find}$$

$$\omega_n = \frac{2\pi}{0.5} = 4\pi = \sqrt{\frac{2.3544\,\rho_0}{40}}$$

Hence $\rho_0 = 2682.8816\ \text{kg/m}^3$.

2.119 Let x = displacement of mass and P = tension in rope on the left of mass. Equations of motion:

$$\sum F = m\,\ddot{x} = -k\,x - P \quad \text{or} \quad P = -m\,\ddot{x} - k\,x \tag{1}$$

$$\sum M = J_0\,\ddot{\theta} = P\, r_2 - c\,(\dot{\theta}\, r_1) \tag{2}$$

Using Eq. (1) in (2), we obtain

$$-(m\,\ddot{x} + k\,x)\, r_2 - c\,\dot{\theta}\, r_1 = J_0\,\ddot{\theta} \tag{3}$$

With $x = \theta\, r_2$, Eq. (3) can be written as:

$$(J_0 + m\, r_2^2)\,\ddot{\theta} + c\, r_1\,\dot{\theta} + k\, r_2^2\,\theta = 0 \tag{4}$$

For given data, Eq. (4) becomes

$$[5 + 10\,(0.25)^2]\,\ddot{\theta} + c\,(0.1)\,\dot{\theta} + k\,(0.25)^2\,\theta = 0$$

$$\text{or} \quad 5.625\,\ddot{\theta} + 0.1\, c\,\dot{\theta} + 0.0625\, k\,\theta = 0 \tag{5}$$

Since amplitude is reduced by 80% in 10 cycles,

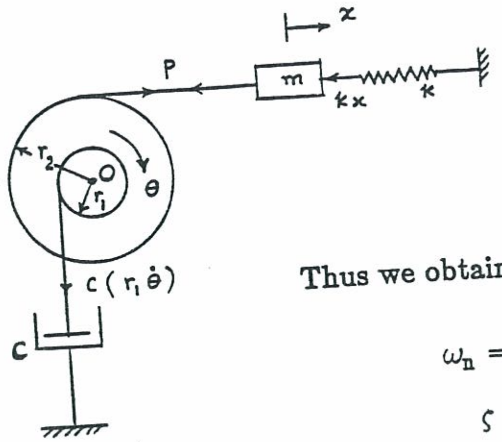
$$\frac{x_1}{x_{11}} = \frac{1.0}{0.2} = 5 = e^{10\,\zeta\,\omega_n\,\tau_d}$$

$$\ln\frac{x_1}{x_{11}} = \ln 5 = 1.6094 = 10\,\zeta\,\omega_n\,\tau_d \tag{6}$$

Since the natural frequency (assumed to be undamped torsional vibration frequency) is 5 Hz, $\omega_n = 2\pi(5) = 31.416$ rad/sec. Also

$$\tau_d = \frac{1}{f_d} = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{0.2}{\sqrt{1-\zeta^2}} \tag{7}$$

Eq. (6) gives



$$1.6094 = 10\,\varsigma\,(31.416)\left(\frac{0.2}{\sqrt{1-\varsigma^2}}\right) = \frac{62.832\,\varsigma}{\sqrt{1-\varsigma^2}}$$

i.e., $\sqrt{1-\varsigma^2} = \frac{62.832}{1.6094}\,\varsigma = 39.0406\,\varsigma$

i.e., $\varsigma = 0.02561$

Thus we obtain:

$$\omega_n = \sqrt{\frac{0.0625\ k}{5.625}} = 31.416 \quad \text{or} \quad k = 8.8827\ (10^4)\ \text{N/m}$$

$$\varsigma = 0.02561 = \frac{c}{c_c} = \frac{c}{2\ m_{eq}\ \omega_n} = \frac{0.10\ c}{2\ (5.625)\ (31.416)}$$

or $c = 90.5134$ N–s/m

2.120 Torque $= 2 \times 10^{-3}$ N-m

angle $= 50^\circ = 80$ divisions

For a torsional system, Eq. (2.84) gives

$$\frac{\theta_1}{\theta_2} = e^{\varsigma\,\omega_n\,\tau_d} \qquad (E_1)$$

(b) For one cycle, $\tau_d = 2$ sec and (E_1) gives

$$\frac{80}{5} = e^{2\,\varsigma\,\omega_n} \quad \text{or} \quad \varsigma\,\omega_n = \frac{1}{2}\ln(16) = 1.3863 \qquad (E_2)$$

Since $\tau_d = \dfrac{2\pi}{\sqrt{\omega_n^2 - \varsigma^2\omega_n^2}}$,

$$\omega_n^2 = \frac{(2\pi)^2}{\tau_d^2} + \varsigma^2\omega_n^2 = \frac{4\pi^2}{4} + 1.3863^2 = 11.7915 \qquad (E_3)$$

i.e., $\omega_n = 3.4339$ rad/sec

(d) Since angular displacement of rotor under applied torque

$= 50^\circ = 0.8727$ rad,

$$k_t = \text{torque/angular displacement} = 2\times10^{-3}/0.8727 \qquad (E_4)$$

$$= 2.2917 \times 10^{-3}\ \text{N-m/rad}$$

(a) Mass moment of inertia of rotor is

$$J_o = \frac{k_t}{\omega_n^2} = 2.2917 \times 10^{-3}/11.7915 = 1.9436 \times 10^{-4}\ \text{N-m-s}^2 \qquad (E_5)$$

$$(E_6)$$

(c) $c_t = 2\,J_o\,\varsigma\,\omega_n$

Eqs. (E_2) and (E_3) give $\varsigma = \dfrac{\varsigma\,\omega_n}{\omega_n} = \dfrac{1.3863}{3.4339} = 0.4037$

Eq. (E_6) gives $c_t = 5.3887 \times 10^{-4}$ N-m-s/rad.

2.121

(a) $m = 10$ kg, $c = 150$ N-s/m, $k = 1000$ N/m

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{10}} = 10 \text{ rad/s}$$

$$\zeta = \frac{c}{2 m \omega_n} = \frac{150}{2(10)(10)} = 0.75$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 10\sqrt{1 - 0.75^2} = 6.61438 \text{ rad/s}$$

(under-damped)

(b) $m = 10$ kg, $c = 200$ N-s/m, $k = 1000$ N/m

$$\omega_n = \sqrt{\frac{k}{m}} = 10 \text{ rad/s}$$

$$\zeta = \frac{c}{2 m \omega_n} = \frac{200}{2(10)(10)} = 1.0$$

$$\omega_d = 10\sqrt{1 - 1.00^2} = 0$$

(critically-damped)

(c) $m = 10$ kg, $c = 250$ N-s/m, $k = 1000$ N/m

$$\omega_n = \sqrt{\frac{k}{m}} = 10 \text{ rad/s}$$

$$\zeta = \frac{c}{2 m \omega_n} = \frac{250}{2(10)(10)} = 1.25$$

ω_d = not applicable (over-damped)

2.122

(a) Underdamped system: Response: Eq. (2.70)

$$X_0 = \left\{ x_0^2 + \left(\frac{\dot{x}_0 + \zeta \omega_n x_0}{\omega_d} \right)^2 \right\}^{1/2} \quad (E.1)$$

Using $x_0 = 0.1$, $\dot{x}_0 = 10$, $\zeta = 0.75$, $\omega_n = 10$, $\omega_d = 6.61438$, Eq. (E.1) gives $X_0 = 1.62832$ m.

$$\phi_0 = \tan^{-1}\left(- \frac{\dot{x}_0 + \zeta \omega_n x_0}{\omega_d x_0} \right)$$

$$= \tan^{-1}\left(- \frac{10 + 0.75(10)(0.1)}{6.61438(0.1)} \right) = -86.47908^\circ = -1.50935 \text{ rad}$$

Eq. (2.70) gives:

$$x(t) = 1.62832\, e^{-7.5t} \cos(6.61438\, t + 1.50935) \text{ m}$$

(b) Critically damped system: Response: Eq. (2.80)

$$x(t) = \{x_0 + (\dot{x}_0 + \omega_n x_0)\, t\}\, e^{-\omega_n t}$$

$$= \{0.1 + (10 + 10 * 0.1)\, t\}\, e^{-10t}$$

$$= (0.1 + 11\, t)\, e^{-10t} \quad m$$

(c) overdamped system: Response: Eq. (2.81)

Using $\sqrt{\zeta^2 - 1} = \sqrt{1.25^2 - 1} = 0.75$, we obtain

$$C_1 = \frac{x_0 \omega_n \{\zeta + \sqrt{\zeta^2 - 1}\} + \dot{x}_0}{2\, \omega_n \sqrt{\zeta^2 - 1}} \qquad \text{Eq. } (2.82)$$

$$= \frac{0.1\,(10)\,\{1.25 + 0.75\} + 10}{2\,(10)\,(0.75)} = 0.8$$

$$C_2 = \frac{-x_0 \omega_n \{\zeta - \sqrt{\zeta^2 - 1}\} - \dot{x}_0}{2\, \omega_n \sqrt{\zeta^2 - 1}} \qquad \text{Eq. } (2.82)$$

$$= \frac{-0.1\,(10)\,\{1.25 - 0.75\} - 10}{2\,(10)\,(0.75)} = -0.7$$

Eq. (2.81) gives

$$x(t) = C_1\, e^{(-\zeta + \sqrt{\zeta^2 - 1})\,\omega_n t} + C_2\, e^{(-\zeta - \sqrt{\zeta^2 - 1})\,\omega_n t}$$

$$= 0.8\, e^{(-1.25 + 0.75)(10)\, t} - 0.7\, e^{(-1.25 - 0.75)(10)\, t}$$

$$= 0.8\, e^{-5t} - 0.7\, e^{-20t} \quad m$$

2.123 Energy dissipated in a cycle of motion, $x(t) = X \sin \omega_d t$, is given by

$$\Delta W = \pi\, c\, \omega_d\, X^2 \qquad (E.1)$$

(a) $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{10}} = 10$ rad/s

$$\zeta = \frac{c}{2 m \omega_n} = \frac{50}{2\,(10)\,(10)} = 0.25$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 10 \sqrt{1 - 0.25^2} = 9.682458 \text{ rad/s}$$

For $X = 0.2$ m, Eq. (E.1) gives

$$\Delta W = \pi (50)(9.682458)(0.2^2) = 60.83682 \text{ Joules}$$

(b) $\omega_n = \sqrt{\frac{k}{m}} = 10$ rad/s

$$\zeta = \frac{c}{2 m \omega_n} = \frac{150}{2(10)(10)} = 0.75$$

$$\omega_d = \omega_n \sqrt{1-\zeta^2} = 10 \sqrt{1-0.75^2} = 6.614378 \text{ rad/s}$$

For $X = 0.2$ m, Eq. (E.1) gives

$$\Delta W = \pi (150)(6.614378)(0.2^2) = 124.678385 \text{ Joules}$$

2.139 $m = 20$ kg , $k = 4000$ N/m

$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4000}{20}} = 14.1421$ rad/sec

Amplitudes of successive cycles : 50, 45, 40, 35 mm

Amplitudes of successive cycles diminish by 5 mm = 5×10^{-3} m

System has Coulomb damping.

$\frac{4 \mu N}{k} = 5 * 10^{-3} \Rightarrow \mu N = \left\{\frac{(5 \times 10^{-3})(4000)}{4}\right\} = 5$ N

= damping force

Frequency of damped vibration = 14.1421 rad/sec.

2.140 $m = 20$ kg, $k = 10000$ N/m , $\frac{4 \mu N}{k} = \frac{150 - 100}{4}$ mm $= 12.5 \times 10^{-3}$ m

$\mu = \frac{(12.5 \times 10^{-3})(10\,000)}{4(20 \times 9.81)} = 0.1593$

Time elapsed $= 4 \tau_n = 4 \times \frac{2\pi}{\omega_n} = 8\pi \sqrt{\frac{m}{k}} = 1.124$ sec

---

2.141 $m = 10$ kg , $k = 3000$ N/m , $\mu = 0.12$, $X = 100$ mm

$\frac{4 \mu N}{k} = \frac{4(0.12)(10 \times 9.81)}{3000} = 0.0157$ m $= 15.7$ mm

As $6\left(\frac{4 \mu N}{k}\right) = 94.2$ mm, mass comes to rest at $(100 - 94.2) = 5.8$ mm

---

2.142 $mg = 25$ N , $k = 1000$ N/m , damping force = constant

Mass released with $x_0 = 10$ cm and $\dot{x}_0 = 0$.

Static deflection of spring due to self weight of mass $= \frac{25}{1000}$

$= 0.025$ m

at $t = 0$: $x = 0.1$ m , $\dot{x} = 0$

$x_0 = 0.1$

$x_1 = x_0 - 2\frac{\mu N}{k}$, $x_2 = x_0 - \frac{4\mu N}{k}$

$x_3 = x_0 - \frac{6\mu N}{k}$, $x_4 = x_0 - \frac{8\mu N}{k} = 0$

i.e., $x_0 = \frac{8\mu N}{k} = 0.1$

Magnitude of damping force $= \mu N = \frac{x_0 k}{8} = \frac{(0.1)(1000)}{8}$
$= 12.5$ N

2.143 $m = 20$ kg, $k = 10{,}000$ N/m , $\mu N = 50$ N , $x_0 = 0.05$ m

(a) Number of half cycles elapsed before mass comes to rest (r) is given by:

$$r \geq \left\{ \frac{x_0 - \frac{\mu N}{k}}{2\frac{\mu N}{k}} \right\} = \frac{0.05 - \left(\frac{50}{10000}\right)}{2\left(\frac{50}{10\,000}\right)} = 4.5$$

$\therefore$ $r = 5$

(b) Time elapsed before mass comes to rest:

$$t_p = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{20}{10000}} = 0.2810 \text{ sec}$$

Time taken $= (2.5 \text{ cycles})\, t_p = 0.7025$ sec

(c) Final extension of spring after 5 half-cycles:

$$x_5 = 0.05 - 5\left(\frac{2\mu N}{k}\right) = 0.05 - 5\left(2 * \frac{50}{10\,000}\right) = 0$$

(displacement from static equilibrium position = 0)

But static deflection $= \frac{mg}{k} = \frac{20 * 9.81}{10\,000} = 0.01962$ m

$\therefore$ Final extension of spring $= 1.9620$ cm.

2.144 (a) Equation of motion for angular oscillations of pendulum:

$$J_0\ddot{\theta} + mgl\sin\theta \pm mg\mu\frac{d}{2}\cos\theta = 0$$

For small angles, $\ddot{\theta} + \frac{mgl}{J_0}\left(\theta \pm \frac{\mu d}{2l}\right) = 0$

This shows that the angle of swing decreases by $\left(\frac{\mu d}{2l}\right)$ in each quarter cycle.

(b) For motion from right to left:

$$\theta(t) = A_1\cos\omega_n t + A_2\sin\omega_n t + \frac{\mu d}{2l}$$

where $\omega_n = \sqrt{\frac{mgl}{J_0}}$.

Let $\theta(t=0) = \theta_0$ and $\dot{\theta}(t=0) = 0$. Then $A_1 = \theta_0 - \frac{\mu d}{2l}$, $A_2 = 0$

$$\theta(t) = \left(\theta_0 - \frac{\mu d}{2l}\right)\cos\omega_n t + \frac{\mu d}{2l}$$

For motion from left to right:

$$\theta(t) = A_3 \cos\omega_n t + A_4 \sin\omega_n t - \frac{\mu d}{2l}$$

At $\omega_n t = \pi$, $\theta = -\theta_0 + \frac{2\mu d}{2l}$, $\dot{\theta} = 0$ from previous solution.

$$A_3 = \theta_0 - \frac{3\mu d}{2l}, \quad A_4 = 0$$

$$\theta(t) = \left(\theta_0 - \frac{3\mu d}{2l}\right)\cos\omega_n t - \frac{\mu d}{2l}$$

(c) The motion ceases when $\left(\theta_0 - n\,\frac{4\mu d}{2l}\right) < \frac{\mu d}{2l}$

where n denotes the number of cycles.

2.145

$x(t) = X \sin\omega t$ (under sinusoidal force $F_0 \sin\omega t$)

Damping force $= \mu N$

Total displacement per cycle $= 4X$

Energy dissipated per cycle $= \Delta W = 4\mu N X$ (E_1)

If c_{eq} = equivalent viscous damping constant, energy dissipated per cycle is given by Eq. (2.98):

$$\Delta W = \pi\, c_{eq}\, \omega X^2 \qquad (E_2)$$

Equating (E_1) and (E_2) gives

$$c_{eq} = \frac{4\mu N X}{\pi\omega X^2} = \frac{4\mu N}{\pi\omega X} \qquad (E_3)$$

2.146

Due to viscous damping:

$$\delta = \ln\left(\frac{X_m}{X_{m+1}}\right) \simeq 2\pi\zeta$$

z_1 = percent decrease in amplitude per cycle at X_m

$$= 100\left(\frac{X_m - X_{m+1}}{X_m}\right) = 100\left(1 - \frac{X_{m+1}}{X_m}\right) = 100\left(1 - e^{-2\pi\zeta}\right)$$

Due to Coulomb damping:

z_2 = percent decrease in amplitude per cycle at X_m

$$= 100\left(\frac{X_m - X_{m+1}}{X_m}\right) = 100\left(\frac{4\mu N}{k X_m}\right)$$

When both types of damping are present:

$$z_1 + z_2\Big|_{X_m = 20\text{ mm}} = 2 \;; \qquad z_1 + z_2\Big|_{X_m = 10\text{ mm}} = 3$$

i.e.,

$$100\left(1-e^{-2\pi\zeta}\right)+\frac{400}{0.02}\left(\frac{\mu N}{k}\right)=2$$

$$100\left(1-e^{-2\pi\zeta}\right)+\frac{400}{0.01}\left(\frac{\mu N}{k}\right)=3$$

The solution of these equations gives

$$50\left(1-e^{-2\pi\zeta}\right)=0.5 \quad \text{and} \quad \frac{\mu N}{k}=0.5\times10^{-6}\ \text{m}$$

2.147 Coulomb damping.

(a) Natural frequency $= \omega_n = \dfrac{2\pi}{\tau_n} = \dfrac{2\pi}{1} = 6.2832$ rad/sec. Reduction in amplitude in each cycle:

$$= \frac{4\,\mu\,N}{k} = 4\,\mu\,g\,\frac{m}{k} = \frac{4\,\mu\,g}{\omega_n^2} = 4\,\mu\left(\frac{9.81}{6.2832^2}\right)$$

$$= 0.9940\,\mu = \frac{0.5}{100} = 0.005\ \text{m}$$

Kinetic coefficient of friction $= \mu = 0.00503$

(b) Number of half-cycles executed (r) is:

$$r \ge \frac{\left(x_0 - \frac{\mu\,N}{k}\right)}{\left(\frac{2\,\mu\,N}{k}\right)} = \frac{\left(x_0 - \frac{\mu\,g}{\omega_n^2}\right)}{\left(\frac{2\,\mu\,g}{\omega_n^2}\right)}$$

$$\ge \frac{\left(0.1 - \frac{0.00503\,(9.81)}{6.2832^2}\right)}{\left(\frac{2\,(0.00503)\,(9.81)}{6.2832^2}\right)}$$

$$\ge 39.5032$$

$$\ge 40$$

Thus the block stops oscillating after 20 cycles.

2.148

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10,000}{5}} = 44.721359 \text{ rad/s}$$

$$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{44.721359} = 0.140497 \text{ s}$$

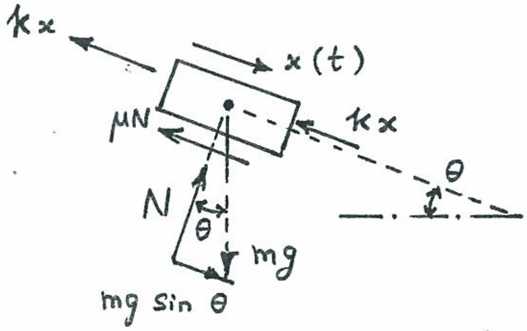
Time taken to complete 10 cycles $= 10\,\tau_n$

$= 1.40497$ s

2.149

(a) $\theta = 30^\circ$

$N = mg \cos\theta$



Case 1: When $x = +$ and $\dot{x} = +$ or $x = -$ and $\dot{x} = +$:

$$m\ddot{x} = -2kx - \mu N + mg\sin\theta$$

or $$m\ddot{x} + 2kx = -\mu mg\cos\theta + mg\sin\theta \qquad (E.1)$$

Case 2: when $x = +$ and $\dot{x} = -$ or $x = -$ and $\dot{x} = -$:

$$m\ddot{x} = -2kx + \mu N + mg\sin\theta$$

or $$m\ddot{x} + 2kx = \mu mg\cos\theta + mg\sin\theta \qquad (E.2)$$

Eqs. (E.1) and (E.2) can be written as a single equation as:

$$m\ddot{x} + \mu mg\cos\theta \;\; sgn(\dot{x}) + 2kx + mg\sin\theta = 0 \qquad (E.3)$$

(b) $x_0 = 0.1$ m , $\dot{x}_0 = 5$ m/s

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{20}} = 7.071068 \text{ rad/s}$$

Solution of Eq. (E.1):

$$x(t) = A_1 \cos\omega_n t + A_2 \sin\omega_n t - \frac{\mu mg\cos\theta}{k} + \frac{mg\sin\theta}{k} \qquad (E.4)$$

Solution of Eq. (E.2):

$$x(t) = A_3 \cos\omega_n t + A_4 \sin\omega_n t + \frac{\mu mg\cos\theta}{k} + \frac{mg\sin\theta}{k} \qquad (E.5)$$

Note: There is no page **2-67.**

Using the initial conditions in each half cycle, the constants A_1 and A_2 or A_3 and A_4 are to be found. For example, in the first half cycle, the motion starts from left toward right with $x_0 = 0.1$ and $\dot{x}_0 = 5$. These values can be used in Eq. (E.4) to find A_1 and A_2.

2.150 Friction force $= \mu N = 0.2\ (5) = 1$ N. $k = \dfrac{25}{0.10} = 250$ N/m. Reduction in amplitude in each cycle $= \dfrac{4\,\mu N}{k} = \dfrac{4\ (1)}{250} = 0.016$ m. Number of half-cycles executed before the motion ceases (r):

$$r \geq \left(\frac{x_0 - \frac{\mu N}{k}}{\frac{2\,\mu N}{k}} \right) = \frac{0.1 - 0.004}{0.008} \geq 12$$

Thus after 6 cycles, the mass stops at a distance of 0.1 - 6 (0.016) = 0.004 m from the unstressed position of the spring.

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{250\ (9.81)}{5}} = 22.1472 \text{ rad/sec}$$

$$\tau_n = \frac{2\,\pi}{\omega_n} = 0.2837 \text{ sec}$$

Thus total time of vibration $= 6\,\tau_n = 1.7022$ sec.

2.151 Energy dissipated in each full load cycle is given by the area enclosed by the hysteresis loop.
The area can be found by counting the squares enclosed by the hysteresis loop. In Fig. 2.117, the number of squares is ≈ 33. Since each square $= \dfrac{100 \times 1}{1000} = 0.1$ N-m, the energy dissipated in a cycle is

$$\Delta W = 33 \times 0.1 = 3.3 \text{ N-m} = \pi\, k\, \beta\, X^2$$

Since the maximum deflection $= X = 4.3$ mm, and the slope of the force-deflection curve is

$$k \simeq \frac{1800 \text{ N}}{11 \text{ mm}} = 1.6364 \times 10^5 \text{ N/m},$$

the hysteresis damping constant β is given by

$$\beta = \frac{\Delta W}{\pi k X^2} = \frac{3.3}{\pi (1.6364 \times 10^5)(0.0043)^2} = 0.3472$$

$\delta = \pi\beta$ = logarithmic decrement $= \pi(0.3472) = 1.0908$

Equivalent viscous damping ratio $= \zeta_{eq} = \beta/2 = 0.1736$.

2.152 $\dfrac{x_j}{x_{j+1}} = \dfrac{2+\pi\beta}{2-\pi\beta} = 1.1$, $\beta = 0.03032$

$c_{eq} = \beta\sqrt{mk} = 0.03032\sqrt{1 \times 2} = 0.04288$ N-s/m

$\Delta W = \pi k \beta X^2 = \pi(2)(0.03032)\left(\frac{10}{1000}\right)^2 = 19.05 \times 10^{-6}$ N-m

---

2.153 Logarithmic decrement $= \delta = \ln\left(\dfrac{x_j}{x_{j+1}}\right) \simeq \pi\beta$

For n cycles, $\delta = \frac{1}{n}\ln\left(\dfrac{x_0}{x_n}\right) \simeq \pi\beta$

$$\frac{1}{100}\ln\left(\frac{30}{20}\right) = 0.004055 = \pi\beta$$

$\beta = 0.001291$

2.154 $\delta = \dfrac{1}{n}\ln\dfrac{X_0}{X_n}$

$$= \frac{1}{100}\ln\frac{25}{10} = \frac{1}{100}\ln 2.5 = 0.0091629$$

$$\delta = \pi\frac{h}{k}$$

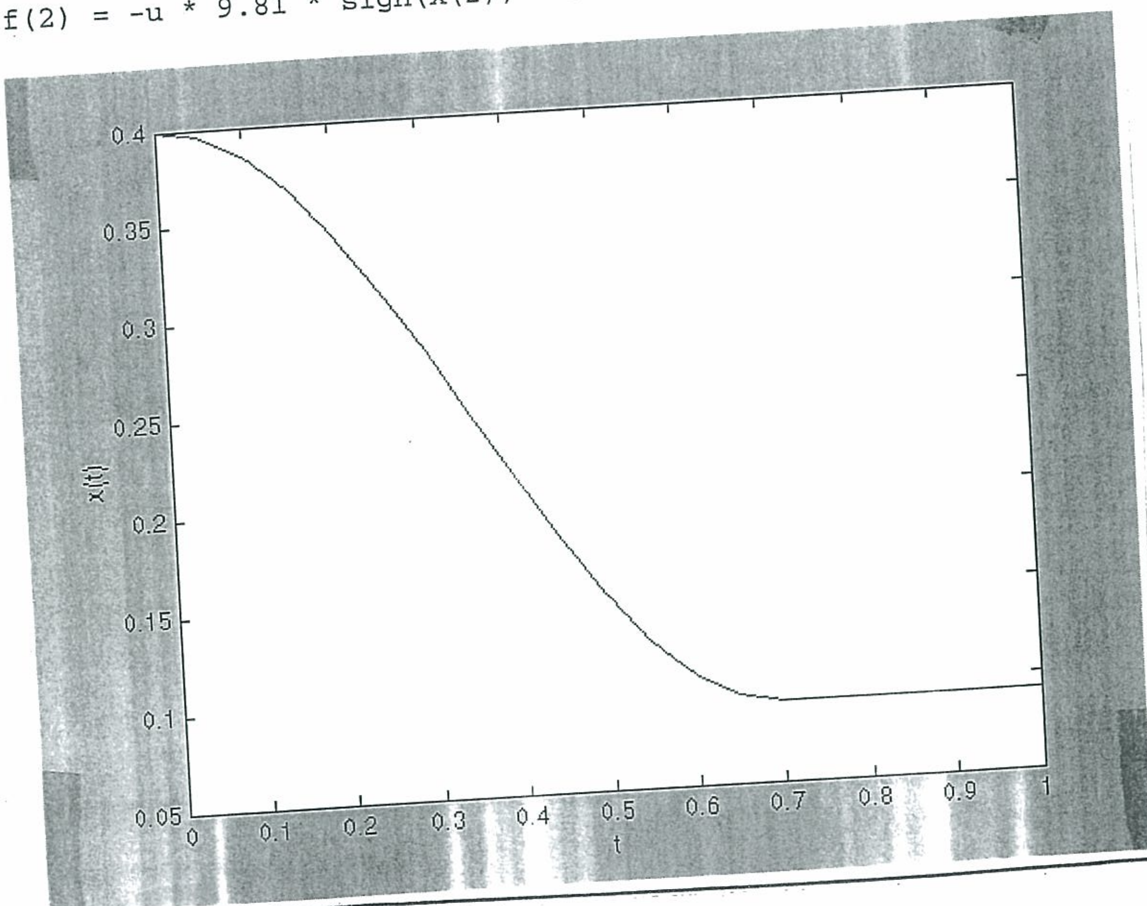
or $h = \dfrac{\delta k}{\pi} = \dfrac{(0.0091629)(200)}{\pi} = 0.583327$ N/m

---

2.157

```
% Ex2_157.m
% This program will use dfunc1.m
tspan = [0: 0.05: 8];
x0 = [0.4; 0.0];
[t, x] = ode23('dfunc1', tspan, x0);
plot(t, x(:, 1));
xlabel('t');
ylabel('x(t)');
```

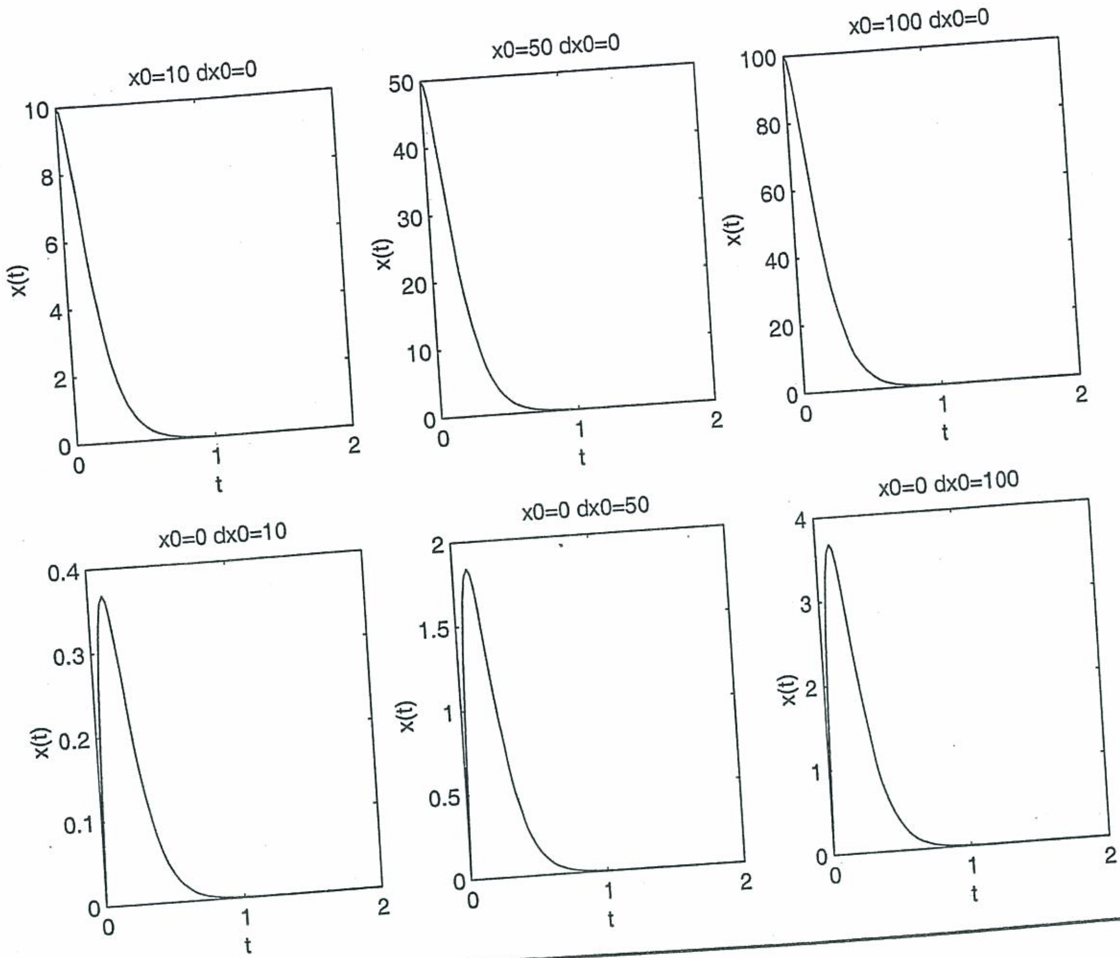
```
% dfunc1.m
function f = dfunc1(t, x)
u = 0.5;
k = 100;
m = 5;
f = zeros(2,1);
f(1) = x(2);
f(2) = -u * 9.81 * sign(x(2)) - k * x(1) / m;
```



2.158

```
% Ex2_158.m
wn = 10;
dx0 = 0;
x0 = 10;
for i = 1:101
    t(i) = 2*(i-1)/100;
    x1(i) = (x0 + ( dx0 + wn*x0)*t(i) )*exp(-wn*t(i));
end
```

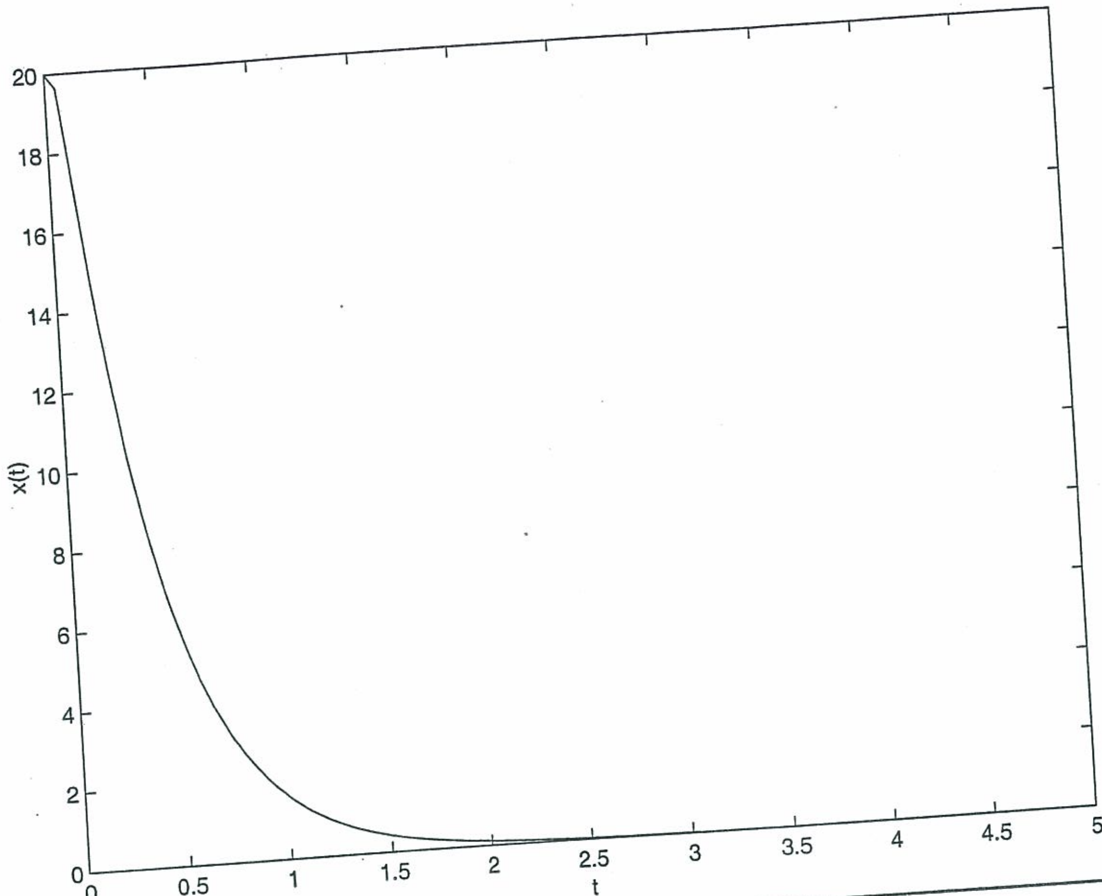
```
x0 = 50;
for i = 1:101
    t(i) = 2*(i-1)/100;
    x2(i) = (x0 + ( dx0 + wn*x0)*t(i) )*exp(-wn*t(i));
end
x0 = 100;
for i = 1:101
    t(i) = 2*(i-1)/100;
    x3(i) = (x0 + ( dx0 + wn*x0)*t(i) )*exp(-wn*t(i));
end
x0 = 0;
dx0 = 10;
for i = 1:101
    t(i) = 2*(i-1)/100;
    x4(i) = (x0 + ( dx0 + wn*x0)*t(i) )*exp(-wn*t(i));
end
dx0 = 50;
for i = 1:101
    t(i) = 2*(i-1)/100;
    x5(i) = (x0 + ( dx0 + wn*x0)*t(i) )*exp(-wn*t(i));
end
dx0 = 100;
for i = 1:101
    t(i) = 2*(i-1)/100;
    x6(i) = (x0 + ( dx0 + wn*x0)*t(i) )*exp(-wn*t(i));
end
subplot(231);
plot(t,x1);
title('x0=10 dx0=0');
xlabel('t');
ylabel('x(t)');
subplot(232);
plot(t,x2);
title('x0=50 dx0=0');
xlabel('t');
ylabel('x(t)');
subplot(233);
plot(t,x3);
title('x0=100 dx0=0');
xlabel('t');
ylabel('x(t)');
subplot(234);
plot(t,x4);
title('x0=0 dx0=10');
xlabel('t');
ylabel('x(t)');
subplot(235);
plot(t,x5);
title('x0=0 dx0=50');
xlabel('t');
ylabel('x(t)');
subplot(236);
plot(t,x6);
title('x0=0 dx0=100');
xlabel('t');
ylabel('x(t)');
```



2.159

```
% Ex2_159.m
wn = 10;
zeta = 2.0;
dx0 = 50;
x0 = 20;
c1 = ( x0*wn*( zeta + sqrt(zeta^2-1) ) + dx0 )/( 2*wn*sqrt(zeta^2-1) );
c2 = ( -x0*wn*( zeta - sqrt(zeta^2-1) ) - dx0 )/( 2*wn*sqrt(zeta^2-1) );
for i = 1:101
    t(i) = 5*(i-1)/100;
    x(i) = c1*exp( (-zeta + sqrt(zeta^2-1)) *wn*t(i) ) ...
        + c2*exp( (-zeta - sqrt(zeta^2-1)) *wn*t(i) );

end
plot(t,x);
xlabel('t');
ylabel('x(t)');
```



2.160

```
Results of Ex2_160.m
****************************
>> program2
Free vibration analysis
of a single degree of freedom analysis

Data:

m=       4.00000000e+000
k=       2.50000000e+003
c=       0.00000000e+000
x0=      1.00000000e+002
xd0=     -1.00000000e+001
n=       50
delt=    1.00000000e-002

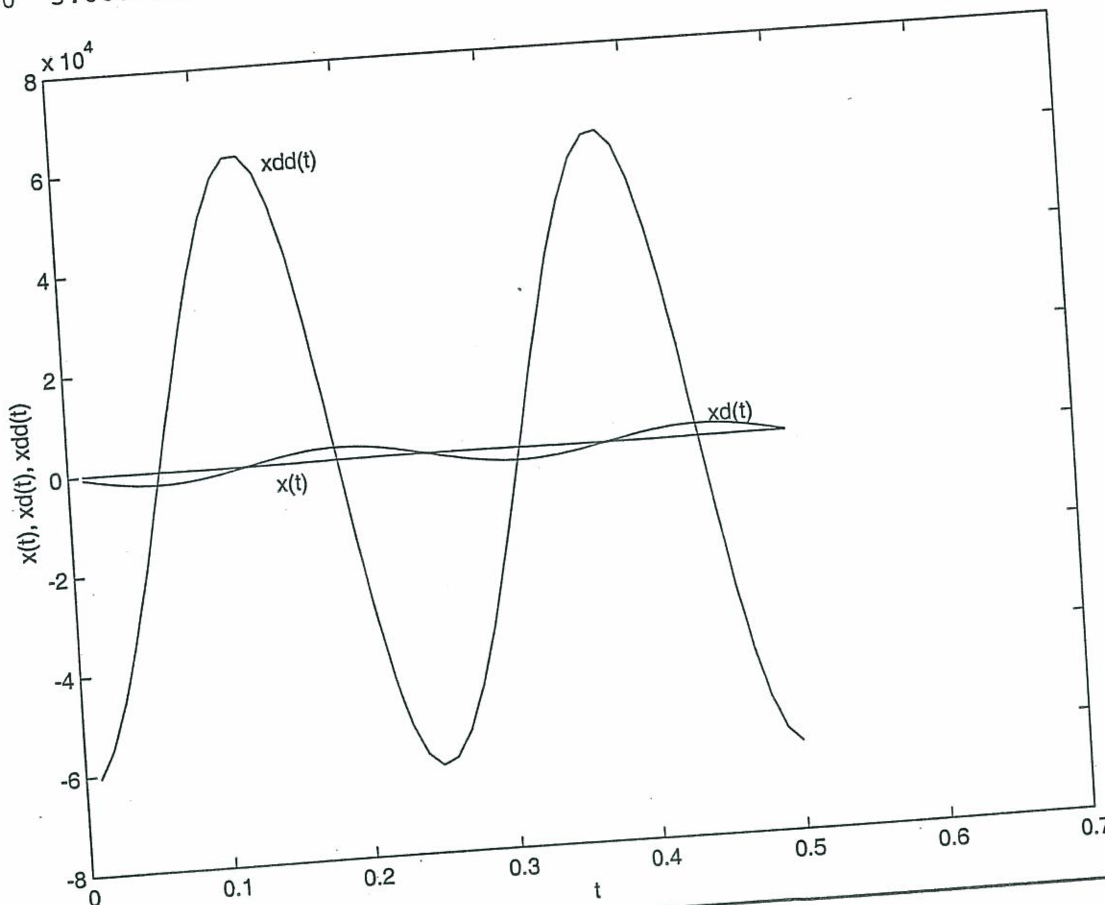
system is undamped

Results:
```

i	time(i)	x(i)	xd(i)	xdd(i)
1	1.000000e-002	9.679228e+001	-6.282079e+002	-6.049518e+004
2	2.000000e-002	8.756649e+001	-1.207348e+003	-5.472905e+004
3	3.000000e-002	7.289623e+001	-1.711420e+003	-4.556014e+004
4	4.000000e-002	5.369364e+001	-2.109085e+003	-3.355853e+004
5	5.000000e-002	3.115264e+001	-2.375618e+003	-1.947040e+004
6	6.000000e-002	6.674722e+000	-2.494445e+003	-4.171701e+003

⋮

44	4.400000e-001	8.425659e-001	2.499931e+003	-5.266037e+002
45	4.500000e-001	2.555609e+001	2.417001e+003	-1.597256e+004
46	4.600000e-001	4.868066e+001	2.183793e+003	-3.042541e+004
47	4.700000e-001	6.877850e+001	1.814807e+003	-4.298656e+004
48	4.800000e-001	8.460003e+001	1.332986e+003	-5.287502e+004
49	4.900000e-001	9.516153e+001	7.682859e+002	-5.947596e+004
50	5.000000e-001	9.980636e+001	1.558176e+002	-6.237897e+004



2.161

```
Results of Ex2_161.m
***************************
>> program2
Free vibration analysis
of a single degree of freedom analysis

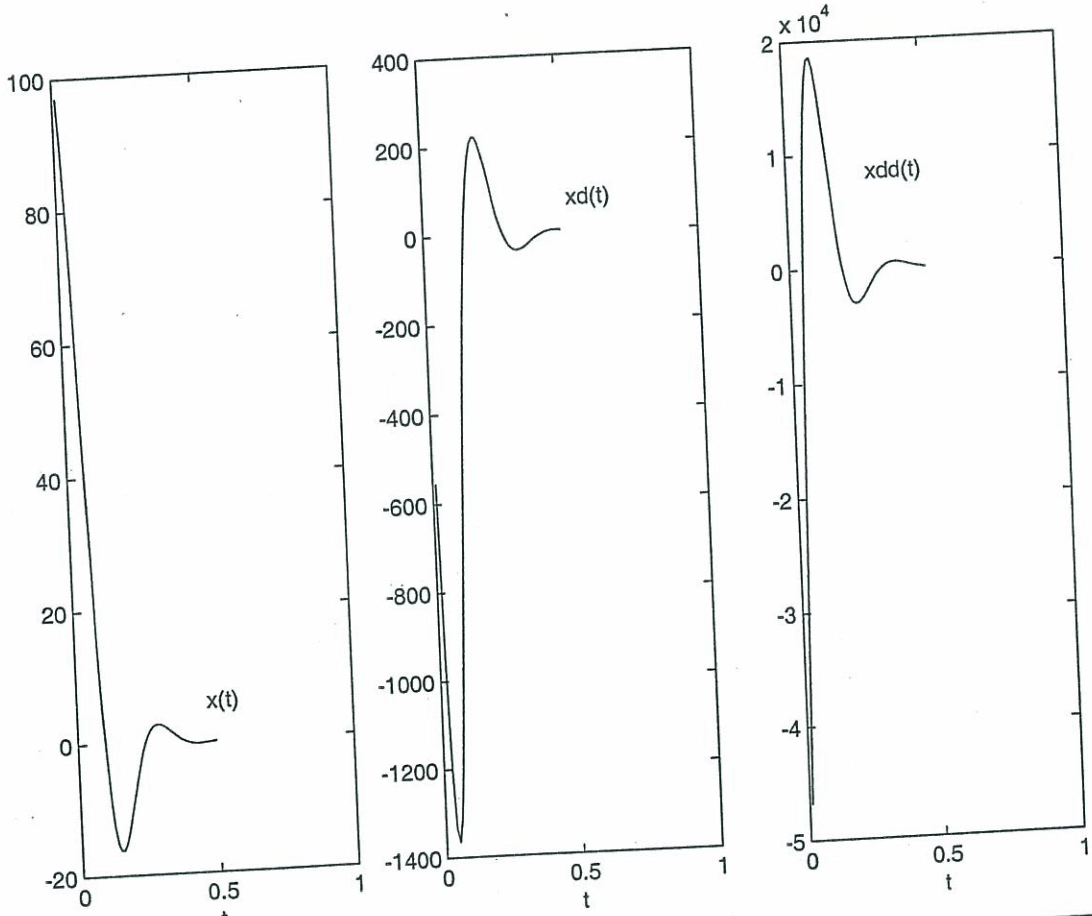
Data:

m=        4.00000000e+000
k=        2.50000000e+003
c=        1.00000000e+002
x0=       1.00000000e+002
xd0=      -1.00000000e+001
n=        50
delt=     1.00000000e-002


system is under damped

Results:
```

i	time(i)	x(i)	xd(i)	xdd(i)
1	1.000000e-002	9.704707e+001	-5.547860e+002	-4.678477e+004
2	2.000000e-002	8.940851e+001	-9.485455e+002	-3.216668e+004
3	3.000000e-002	7.854100e+001	-1.203024e+003	-1.901253e+004
4	4.000000e-002	6.575661e+001	-1.335030e+003	-7.722135e+003
5	5.000000e-002	5.218268e+001	-1.364393e+003	1.495649e+003
6	6.000000e-002	3.874058e+001	-1.312202e+003	8.592187e+003
⋮				
45	4.500000e-001	-4.071590e-001	3.283084e+000	1.723973e+002
46	4.600000e-001	-3.667451e-001	4.698554e+000	1.117518e+002
47	4.700000e-001	-3.150951e-001	5.542443e+000	5.837337e+001
48	4.800000e-001	-2.575358e-001	5.894760e+000	1.359090e+001
49	4.900000e-001	-1.985409e-001	5.844858e+000	-2.203340e+001
50	5.000000e-001	-1.416733e-001	5.484453e+000	-4.856551e+001



2.162

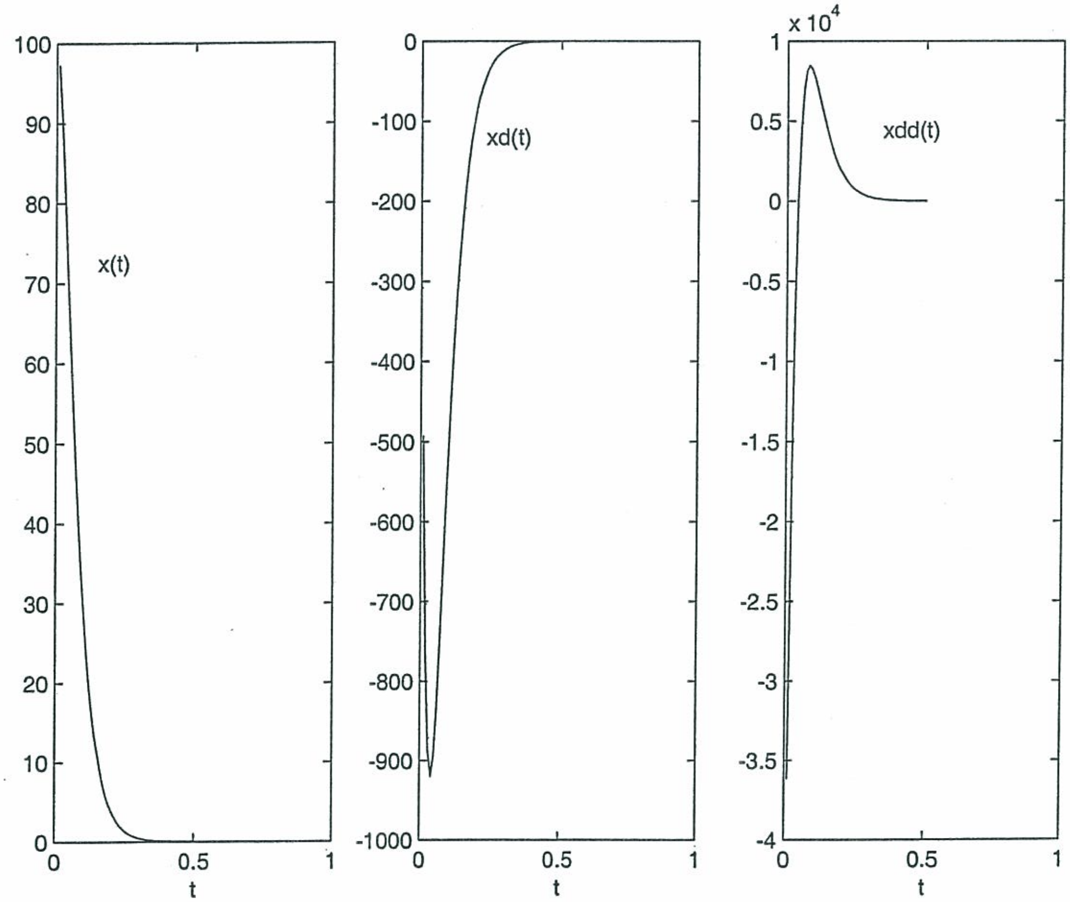
```
Results of Ex2_162.m
****************************
>> program2
Free vibration analysis
of a single degree of freedom analysis

Data:

m=       4.00000000e+000
k=       2.50000000e+003
c=       2.00000000e+002
x0=      1.00000000e+002
xd0=     -1.00000000e+001
n=       50
delt=    1.00000000e-002

system is critically damped

Results:
```



i	time(i)	x(i)	xd(i)	xdd(i)
1	1.000000e-002	9.727222e+001	-4.925915e+002	-3.616556e+004
2	2.000000e-002	9.085829e+001	-7.611960e+002	-1.872663e+004
3	3.000000e-002	8.252244e+001	-8.868682e+002	-7.233113e+003
4	4.000000e-002	7.342874e+001	-9.196986e+002	9.196986e+001
5	5.000000e-002	6.432033e+001	-8.946112e+002	4.530357e+003

•
•
•

44	4.400000e-001	1.996855e-002	-4.576266e-001	1.040098e+001
45	4.500000e-001	1.587541e-002	-3.644970e-001	8.302721e+000
46	4.600000e-001	1.261602e-002	-2.901765e-001	6.623815e+000
47	4.700000e-001	1.002181e-002	-2.309008e-001	5.281410e+000
48	4.800000e-001	7.957984e-003	-1.836505e-001	4.208785e+000
49	4.900000e-001	6.316833e-003	-1.460059e-001	3.352274e+000
50	5.000000e-001	5.012349e-003	-1.160293e-001	2.668750e+000

2.163

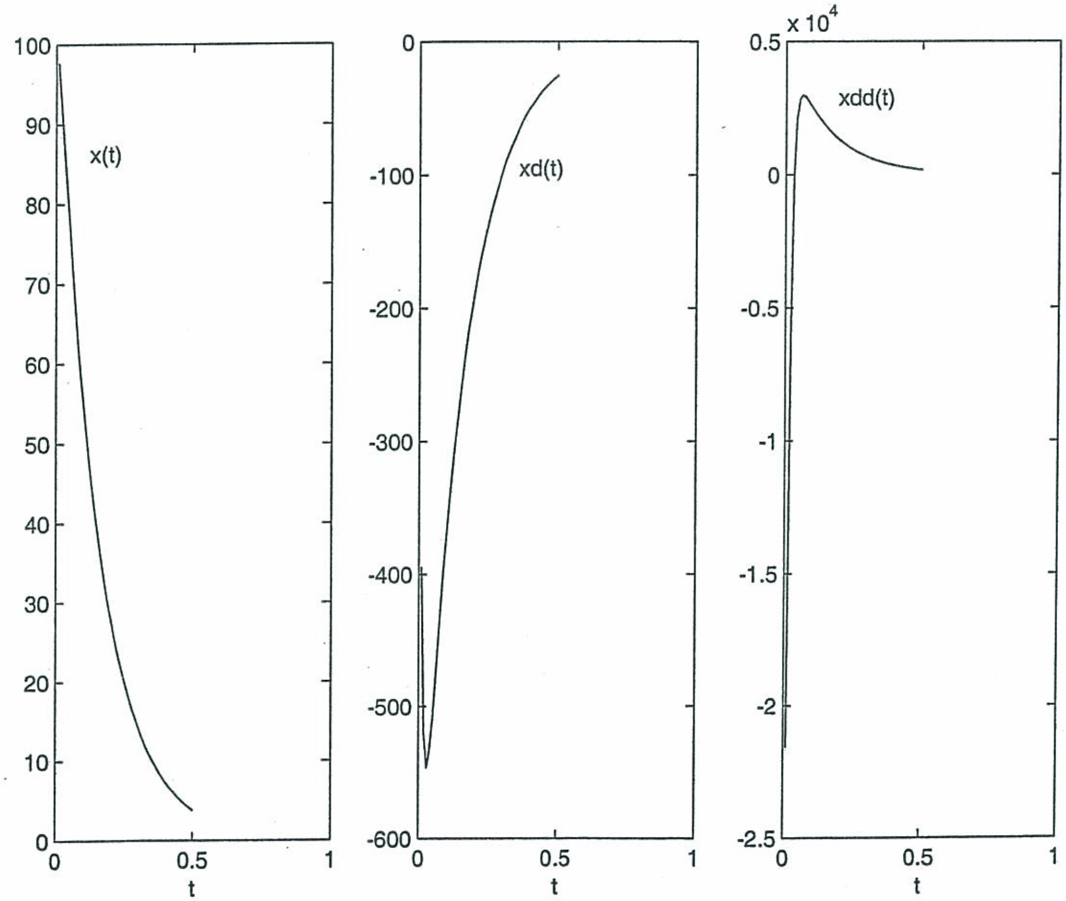
```
Results of Ex2_163.m
*************************
>> program2
Free vibration analysis
of a single degree of freedom analysis

Data:

m=       4.00000000e+000
k=       2.50000000e+003
c=       4.00000000e+002
x0=      1.00000000e+002
xd0=     -1.00000000e+001
n=       50
delt=    1.00000000e-002


system is over damped

Results:
```

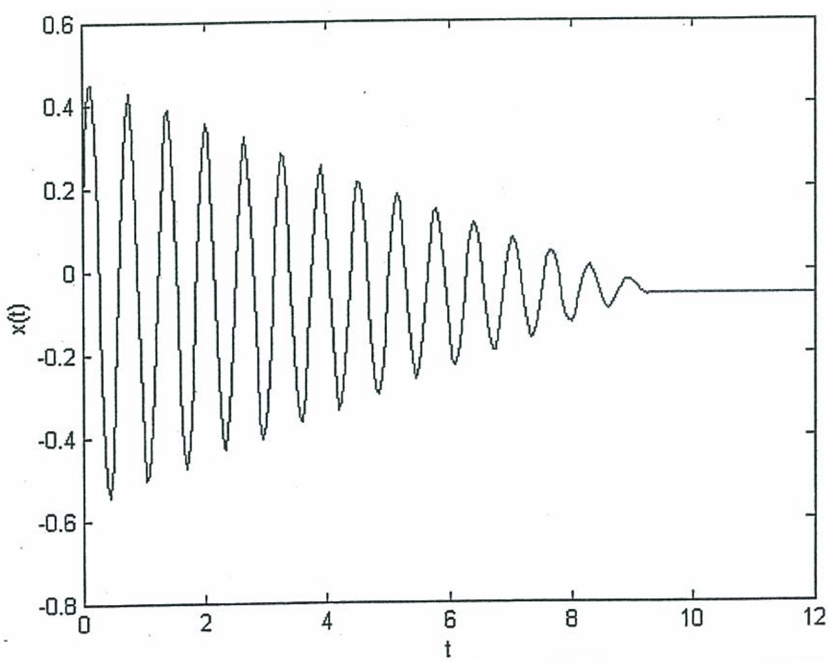


i	time(i)	x(i)	xd(i)	xdd(i)
1	1.000000e-002	9.764929e+001	-3.945541e+002	-2.157540e+004
2	2.000000e-002	9.294636e+001	-5.205155e+002	-6.039927e+003
3	3.000000e-002	8.756294e+001	-5.463949e+002	-8.734340e+001
4	4.000000e-002	8.214078e+001	-5.344391e+002	2.105923e+003
5	5.000000e-002	7.691749e+001	-5.090344e+002	2.830006e+003
⋮				
45	4.500000e-001	5.281309e+000	-3.537806e+001	2.369881e+002
46	4.600000e-001	4.939118e+000	-3.308581e+001	2.216329e+002
47	4.700000e-001	4.619098e+000	-3.094209e+001	2.072727e+002
48	4.800000e-001	4.319813e+000	-2.893726e+001	1.938429e+002
49	4.900000e-001	4.039920e+000	-2.706233e+001	1.812832e+002
50	5.000000e-001	3.778161e+000	-2.530888e+001	1.695374e+002

2.164

```
% Ex2_164.m
% This program will use dfunc2_164.m
tspan = [0: 0.05: 12];
x0 = [0.1; 5];
[t, x] = ode23('dfunc2_164', tspan, x0);
plot(t, x(:, 1));
xlabel('t');
ylabel('x(t)');

% dfunc2_134.m
function f = dfunc2_164(t, x)
u = 0.1;
k = 1000;
m = 20;
g = 9.81;
theta = 30 * pi/180;
f = zeros(2,1);
f(1) = x(2);
f(2) = -u*g*cos(theta)*sign(x(2)) - 2*k*x(1)/m - g*sin(theta);
```



2.165 The equations for the natural frequencies of vibration were derived in Problem 2.35.

Operating speed of turbine is:

$$\omega_0 = (2400)\frac{2\pi}{60} = 251.328 \text{ rad/sec}$$

Thus we need to satisfy:

$$\omega_n\Big|_{axial} = \left\{\frac{g\, l\, A E}{W a (l-a)}\right\}^{1/2} \geq \omega_0 \qquad (E_1)$$

$$\omega_n\Big|_{transverse} = \left\{\frac{3 E I\, l^3 g}{W a^3 (l-a)^3}\right\}^{1/2} \geq \omega_0 \qquad (E_2)$$

$$\omega_n\Big|_{circumferential} = \left\{\frac{G J}{J_0}\left(\frac{1}{a} + \frac{1}{l-a}\right)\right\}^{1/2} \geq \omega_0 \qquad (E_3)$$

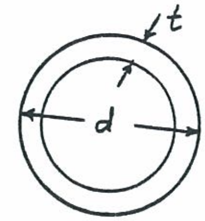
where $A = \frac{\pi d^2}{4}$, $W = 1000 \times 9.81 = 9810$ N,

$I = \frac{\pi d^4}{64}$, $J = \frac{\pi d^2}{32}$, $J_0 = 500$ kg-m^2,

and $E = 207 \times 10^9$ N/m^2, $G = 79.3 \times 10^9$ N/m^2 (for steel).

The unknowns d, l and a can be determined to satisfy the inequalities (E_1), (E_2) and (E_3) using a trial and error procedure.

2.166 From solution of problem 2.38, the requirements can be stated as:



$$\omega_n \Big|_{\text{pivot ends}} = \sqrt{\frac{12\,E\,I}{\ell^3\left(\frac{W}{g} + m_{eff\,1}\right)}} \geq \omega_0 \qquad (E_1)$$

where $E = 30 \times 10^6$ psi and $I = \frac{\pi}{64}\left[d^4 - (d-2t)^4\right]$

$$\omega_n \Big|_{\text{fixed ends}} = \sqrt{\frac{48\,E\,I}{\ell^3\left(\frac{W}{g} + m_{eff\,2}\right)}} \geq \omega_0 \qquad (E_2)$$

with $m_{eff1} = (0.2357\ m)$, $m_{eff2} = (0.3714\ m)$,

$m =$ mass of each column $= \frac{\pi}{4}\left[d^2 - (d-2t)^2\right]\frac{\ell\,\rho}{g}$,

$\rho = 0.283$ lb/in^3, $g = 386.4$ in/sec^2,

$\ell =$ length of column $= 96$ in.,

$W =$ weight of floor $= 4000$ lb,

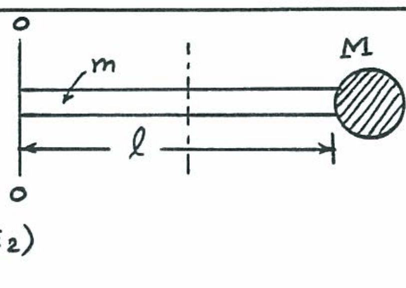
$$\underset{\sim}{W} = \text{weight of columns} = 4\left\{\frac{\pi}{4}\left[d^2 - (d-2t)^2\right]\ell\,\rho\right\} \qquad (E_3)$$

Frequency limit $= \omega_0 = 50 \times 2\pi = 314.16$ rad/sec.

Problem: Find d and t such that $\underset{\sim}{W}$ given by Eq. (E_3) is minimized while satisfying the inequalities (E_1) and (E_2).

This problem can be solved either by graphical optimization or by using a trial and error procedure.

2.167 $J_0 = \frac{m\ell^2}{12} + \frac{m\ell^2}{4} + M\ell^2 = \frac{1}{3}m\ell^2 + M\ell^2 \quad \text{--- } (E_1)$



(i) Viscous damping:

$$\omega_n = \sqrt{\frac{k_t}{J_0}} = \left(\frac{k_t}{\frac{1}{3}m\ell^2 + M\ell^2}\right)^{\frac{1}{2}} \quad \text{--- } (E_2)$$

$$(c_t)_{cri} = 2\,J_0\,\omega_n = 2\sqrt{J_0\,k_t} \quad \text{--- } (E_3)$$

For critical damping, Eq. (2.80) gives

$$\theta(t) = \left\{\theta_0 + (\dot{\theta}_0 + \omega_n\,\theta_0)\,t\right\} e^{-\omega_n t} \quad \text{--- } (E_4)$$

For $\theta_0 = 75^\circ = 1.309$ rad and $\dot{\theta}_0 = 0$,

$$\theta(t) = \left(1.309 + 1.309\, \omega_n t\right) e^{-\omega_n t} \qquad \text{--- } (E_5)$$

For $\theta = 5^\circ = 0.08727$ rad, Eq. (E_5) becomes

$$0.08727 = 1.309\,(1 + \omega_n t)\, e^{-\omega_n t} \qquad \text{--- } (E_6)$$

Let time to return = 2 sec. Then Eq. (E_6) gives

$$0.08727 = 1.309\left(1 + 2\omega_n\right) e^{-2\omega_n} \qquad \text{--- } (E_7)$$

Solve (E_7) by trial and error to find ω_n. Then choose the values of m, M and k_t to get the desired value of ω_n. Find the damping constant $(c_t)_{cri}$ using Eq. (E_3).

(ii) Coulomb damping:

(a) Follow the procedure of part(i) to find the value of ω_n.

(b) Derive expression for the equivalent torsional viscous damping constant $(c_t)_{eq}$ for Coulomb damping. This expression, for small amounts of damping, is

$$(c_t)_{eq} = \left\{ 4\, T_d / \pi\, \omega_n\, \Theta \right\} \qquad \text{--- } (E_8)$$

where T_d = friction (damping) torque, and
Θ = amplitude of angular oscillations.

(c) If $(c_t)_{eq}$ is to be equal to $(c_t)_{cri} = 2\sqrt{J_0\, k_t}$, we find

$$T_d = \frac{\pi\, \omega\, \Theta}{4}\left(2\sqrt{J_0\, k_t}\right) \qquad \text{--- } (E_9)$$

2.168 Let x = vertical displacement of the mass (lunar excursion module), x_s = resulting deflection of each inclined leg (spring). From equivalence of potential energy, we find:
k_{eq_1} = stiffness of each leg in vertical direction = $k \cos^2 \alpha$
Hence for the four legs, the equivalent stiffness in vertical direction is:

$$k_{eq} = 4\, k \cos^2 \alpha$$

Similarly, the equivalent damping coefficient of the four legs in vertical direction is:

$$c_{eq} = 4\, c \cos^2 \alpha$$

where c = damping constant of each leg (in axial motion). Modeling the system as a single degree of freedom system, the equation of motion is:

$$m_{eq}\, \ddot{x} + c_{eq}\, \dot{x} + k_{eq}\, x = 0$$

and the damped period of vibration is:

$$\tau_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n \sqrt{1-\zeta^2}} = \frac{2\pi}{\sqrt{\dfrac{k_{eq}}{m_{eq}}}\sqrt{1-\left(\dfrac{c_{eq}^2}{4\, k_{eq}\, m_{eq}}\right)}}$$

Using $m_{eq} = 2000$ kg, $k_{eq} = 4\, k \cos^2 \alpha$, $c_{eq} = 4\, c \cos^2 \alpha$, and $\alpha = 20°$, the values of k and c can be determined (by trial and error) so as to achieve a value of τ_d between 1 s and 2 s. Once k and c are known, the spring (helical) and damper (viscous) can be designed suitably.

2.169 Assume no damping. Neglect masses of telescoping boom and strut. Find stiffness of telescoping boom in vertical direction (see Example 2.4). Find the equivalent stiffness of telescoping boom together with the strut in vertical direction. Model the system as a single degree of freedom system with natural time period:

$$\tau_n = \frac{2\pi}{\omega_n} = 2\pi \sqrt{\frac{m_{eq}}{k_{eq}}}$$

Using $\tau_n = 1$ s and $m_{eq} = \left(\dfrac{W_c + W_f}{g}\right) = \dfrac{300}{386.4}$, determine the axial stiffness of the strut (k_s). Once k_s is known, the cross section of the strut (A_s) can be found from:

$$k_s = \frac{A_s\, E_s}{\ell_s}$$

with $E_s = 30\,(10^6)$ psi and ℓ_s = length of strut (known).

Chapter 3

Harmonically Excited Vibration

3.1 (a) $\delta = \frac{W}{k} = \frac{50}{4000} = 0.0125$ m

(b) $\delta_{st} = \frac{F_0}{k} = \frac{60}{4000} = 0.015$ m

(c) $\omega_n = \sqrt{\frac{k}{m}} = \left(\frac{4000 \times 9.81}{50}\right)^{1/2} = 28.0143$ rad/sec

$\omega = 6$ Hz $= 37.6992$ rad/sec

$$X = \delta_{st} \left| \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2} \right| = 0.015 \left| \frac{1}{1-\left(\frac{37.6992}{28.0143}\right)^2} \right| = 0.0185 \text{ m}$$

3.2 $$\tau_b = \frac{2\pi}{\omega_n - \omega} = \frac{2\pi}{2\pi\,(40.0 - 39.8)} = 5 \text{ sec}$$

3.3 $k = 4000$ N/m, $m = 10$ kg, $F(t) = 400 \cos 10t$ N

$F_0 = 400$ N, $\omega = 10$ rad/s

$\omega_n = \sqrt{\frac{k}{m}} = 20$ rad/s, $\frac{\omega}{\omega_n} = \frac{10}{20} = 0.5 < 1$

Response is given by Eq. (3.9):

$$x(t) = \left(x_0 - \frac{F_0}{k - m\omega^2}\right)\cos\omega_n t + \left(\frac{\dot{x}_0}{\omega_n}\right)\sin\omega_n t + \frac{F_0}{k - m\omega^2}\cos\omega t \qquad (E.1)$$

(a) $x_0 = 0.1$, $\dot{x}_0 = 0$:

Eq. (E.1) becomes

$$x(t) = \left\{0.1 - \frac{400}{4000 - 10(100)}\right\}\cos 20t + \frac{400}{4000 - 10(100)}\cos 10t$$

$$= -0.033333 \cos 20t + 0.133333 \cos 10t \qquad (E.2)$$

(b) $x_0 = 0$, $\dot{x}_0 = 10$:

Eq. (E.1) becomes

$$x(t) = \left\{0 - \frac{400}{4000 - 10(100)}\right\}\cos 20t + \frac{10}{20}\sin 20t$$

$$+ \left\{\frac{400}{4000 - 10(100)}\right\}\cos 10t$$

$$= -0.133333 \cos 20t + 0.5 \sin 20t + 0.133333 \cos 10t \qquad (E.3)$$

(c) $x_0 = 0.1, \quad \dot{x}_0 = 10$:

Eq. (E.1) becomes

$$x(t) = \left\{0.1 - \frac{400}{4000 - 10(100)}\right\} \cos 20t + \frac{10}{20} \sin 20t$$

$$+ \left\{\frac{400}{4000 - 10(100)}\right\} \cos 10t$$

$$= -0.033333 \cos 20t + 0.5 \sin 20t + 0.133333 \cos 10t \qquad (E.4)$$

(3.4) $k = 4000$ N/m, $m = 10$ kg, $F(t) = 400 \cos 20t$ N,

$F_0 = 400$ N, $\omega = 20$ rad/s

$\omega_n = \sqrt{\frac{k}{m}} = 20$ rad/s, $\quad \frac{\omega}{\omega_n} = \frac{20}{20} = 1$

Response is given by Eq. (3.15):

$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \frac{\delta_{st}\, \omega_n t}{2} \sin \omega_n t \qquad (E.1)$$

where $\delta_{st} = F_0/k = 400/4000 = 0.1$

(a) $x_0 = 0.1, \quad \dot{x}_0 = 0$:

Eq. (E.1) gives

$$x(t) = 0.1 \cos 20t + \frac{(0.1)(20)\,t}{2} \sin 20t$$

$$= 0.1 \cos 20t + t \sin 20t \qquad (E.2)$$

(b) $x_0 = 0, \quad \dot{x}_0 = 10$:

Eq. (E.1) gives

$$x(t) = \frac{10}{20} \sin 20t + \frac{(0.1)(20)\,t}{2} \sin 20t$$

$$= 0.5 \sin 20t + t \sin 20t \qquad (E.3)$$

(c) $x_0 = 0.1, \quad \dot{x}_0 = 10$:

Eq. (E.1) gives

$$x(t) = 0.1 \cos 20t + \frac{10}{20} \sin 20t + \frac{(0.1)(20)\,t}{2} \sin 20t$$

$$= 0.1 \cos 20t + 0.5 \sin 20t + t \sin 20t \qquad (E.4)$$

3.5 $k = 4000$ N/m, $m = 10$ kg, $F(t) = 400 \cos 20.1 t$ N

$F_0 = 400$ N, $\omega = 20.1$ rad/s, $\omega^2 = 404.01$ (rad/s)^2

$\omega_n = \sqrt{\frac{k}{m}} = 20$ rad/s

Solution is given by Eq. (3.9):

$$x(t) = \left(x_0 - \frac{F_0}{k - m\omega^2}\right) \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t$$

$$+ \left(\frac{F_0}{k - m\omega^2}\right) \cos \omega t \qquad (E.1)$$

(a) $x_0 = 0.1$, $\dot{x}_0 = 0$:

Eq. (E.1) reduces to

$$x(t) = \left\{0.1 - \frac{400}{4000 - 10(404.01)}\right\} \cos 20 t$$

$$+ \left\{\frac{400}{4000 - 10(404.01)}\right\} \cos 20.1 t$$

$$= 10.075062 \cos 20 t - 9.975062 \cos 20.1 t \qquad (E.2)$$

(b) $x_0 = 0$, $\dot{x}_0 = 10$:

Eq. (E.1) reduces to

$$x(t) = -\left\{\frac{400}{4000 - 10(404.01)}\right\} \cos 20 t + \frac{10}{20} \sin 20 t$$

$$+ \left\{\frac{400}{4000 - 10(404.01)}\right\} \cos 20.1 t$$

$$= 9.975062 \cos 20 t + 0.5 \sin 20 t$$

$$- 9.975062 \cos 20.1 t \qquad (E.3)$$

(c) $x_0 = 0.1$, $\dot{x}_0 = 10$:

Eq. (E.1) gives

$$x(t) = \left\{0.1 - \frac{400}{4000 - 10(404.01)}\right\} \cos 20 t + \frac{10}{20} \sin 20 t$$

$$+ \left\{\frac{400}{4000 - 10(404.01)}\right\} \cos 20.1 t$$

$$= 10.075062 \cos 20t + 0.5 \sin 20t - 9.975062 \cos 20.1t \qquad (E.4)$$

3.6 $k = 4000$ N/m, $m = 10$ kg, $F(t) = 400 \cos 30t$ N

$F_0 = 400$ N, $\omega = 30$ rad/s, $\omega^2 = 900$ (rad/s)^2

$\omega_n = \sqrt{\frac{k}{m}} = 20$ rad/s, $\frac{\omega}{\omega_n} = 1.5 > 1$

Solution is given by Eq. (3.9):

$$x(t) = \left(x_0 - \frac{F_0}{k - m\omega^2}\right) \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \left(\frac{F_0}{k - m\omega^2}\right) \cos \omega t \qquad (E.1)$$

(a) $x_0 = 0.1$, $\dot{x}_0 = 0$:

Eq. (E.1) yields

$$x(t) = \left\{0.1 - \frac{400}{4000 - 10(900)}\right\} \cos 20t + \frac{400}{4000 - 10(900)} \cos 30t$$

$$= 0.18 \cos 20t - 0.08 \cos 30t \qquad (E.2)$$

(b) $x_0 = 0$, $\dot{x}_0 = 10$:

Eq. (E.1) yields:

$$x(t) = -\left(\frac{400}{4000 - 10(900)}\right) \cos 20t + \frac{10}{20} \sin 20t + \left(\frac{400}{4000 - 10(900)}\right) \cos 30t$$

$$= 0.08 \cos 20t + 0.5 \sin 20t - 0.08 \cos 30t \qquad (E.3)$$

(c) $x_0 = 0.1$, $\dot{x}_0 = 10$:

Eq. (E.1) yields

$$x(t) = \left\{0.1 - \frac{400}{4000 - 10(900)}\right\} \cos 20t + \frac{10}{20} \sin 20t + \left\{\frac{400}{4000 - 10(900)}\right\} \cos 30t$$

$$= 0.18 \cos 20t + 0.5 \sin 20t - 0.08 \cos 30t \qquad (E.4)$$

3.7 $\delta_{st} = \frac{F_0}{k} = \frac{25}{2000} = 0.0125$ m

steady state solution at resonance $= x(t) = \frac{\delta_{st} \cdot \omega_n t}{2} \sin \omega_n t$

$= 0.00625\ \omega_n t \sin \omega_n t$ m

(a) At end of $\frac{1}{4}$ cycle, $\omega_n t = \frac{\pi}{2}$ and $x(t) = 0.00625\left(\frac{\pi}{2}\right) \sin \frac{\pi}{2} = 0.009817$ m

(b) At end of $2\frac{1}{2}$ cycles, $\omega_n t = 5\pi$ and $x(t) = 0.00625(5\pi) \sin 5\pi = 0$

(c) At end of $5\frac{3}{4}$ cycles, $\omega_n t = 11\frac{1}{2}\pi$ and

$x(t) = 0.00625\left(\frac{23}{2}\pi\right) \sin \frac{23}{2}\pi = -0.2258$ m

---

3.8 $\delta_{st} = \frac{F_0}{k} = \frac{100}{4000} = 0.025$ m.

$X = \delta_{st} \cdot \frac{1}{\left|1 - \left(\frac{\omega}{\omega_n}\right)^2\right|}$, $\left|1 - \left(\frac{\omega}{\omega_n}\right)^2\right| = \frac{\delta_{st}}{X} = \frac{0.025}{20 \times 10^{-3}} = 1.25$

$\frac{\omega}{\omega_n} = \sqrt{1.25 + 1} = 1.5$

$\omega_n = \omega/1.5 = 5(2\pi)/1.5 = 20.944$ rad/sec

$m = k/\omega_n^2 = 4000/(20.944)^2 = 9.1189$ kg

3.9 $\omega_n = \sqrt{k/m} = \sqrt{5000/10} = 22.3607$ rad/sec

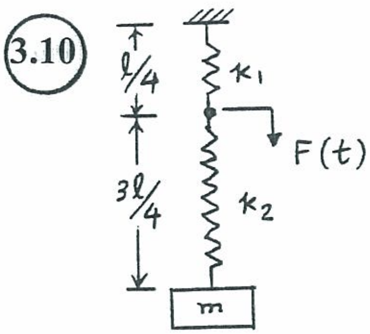
$\delta_{st} = F_0/k = 250/5000 = 0.05$ m

$X = \delta_{st} \left\{ \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \right\}$

i.e., $\omega = \omega_n \left(1 - \frac{\delta_{st}}{X}\right)^{\frac{1}{2}} = 22.3607 \left[1 - \frac{0.05}{0.10}\right]^{\frac{1}{2}}$

$= 15.8114$ rad/sec

3.10



$k_1 = 4k$; $\frac{1}{4k} + \frac{1}{k_2} = \frac{1}{k}$, $k_2 = \frac{4}{3}k$

Force transmitted to the mass through k_2:

$$\tilde{F}(t) = \frac{k_2}{k_1 + k_2} F(t) = \frac{k_1 k_2}{k_1 + k_2}\left(\frac{F_0}{k_1}\right)\cos\omega t$$

$$= k\,\delta_{st}\cos\omega t \quad \text{where} \quad \delta_{st} = \frac{F_0}{k_1}$$

Steady state response of m:

$$x(t) = \frac{\tilde{F}_0}{k\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}}\cos\omega t$$

$$= \left\{\frac{\delta_{st}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right\}\cos\omega t \quad \text{with} \quad \tilde{F}_0 = k\cdot\delta_{st}.$$

3.16 Equivalent stiffness of wing (beam) at location of engine:

$$k = \frac{\text{force}}{\text{deflection}} = \frac{3\,E\,I}{\ell^3} = \frac{3\,E\,(\frac{1}{12}\,b\,a^3)}{\ell^3} = \frac{E\,b\,a^3}{4\,\ell^3}$$

Magnitude of unbalanced force: $= m\,r\,\omega^2 = m\,r\left(\frac{2\,\pi\,N}{60}\right)^2 = \frac{m\,r\,\pi^2\,N^2}{900}$

Equivalent mass of wing at location of engine: $M = \frac{33}{140}\,m_w = \frac{33}{140}\,(a\,b\,\ell\,\rho)$

Equation of motion: $M\,\ddot{x} + k\,x = m\,r\,\omega^2 \sin\omega\,t$

Maximum steady state displacement of wing at location of engine:

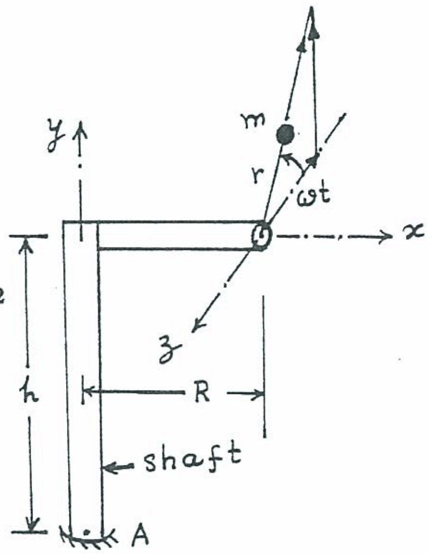
$$X = \left|\frac{m\,r\,\omega^2}{k - M\,\omega^2}\right| = \left|\frac{\left(\frac{m\,r\,\pi^2\,N^2}{900}\right)}{\left\{\frac{E\,b\,a^3}{4\,\ell^3} - \frac{33}{140}\,a\,b\,\ell\,\rho\left(\frac{2\,\pi\,N}{60}\right)^2\right\}}\right|$$

$$= \left|\frac{m\,r\,\ell^3\,N^2}{22.7973\,E\,b\,a^3 - 0.2357\,\rho\,a\,b\,\ell^4\,N^2}\right|$$

3.17 Rotating unbalanced force, $m\, r\, \omega^2$, can be resolved into two components as:

$$F_y = m\, r\, \omega^2 \sin \omega t \quad \text{(parallel y—axis)}$$
$$F_z = m\, r\, \omega^2 \cos \omega t \quad \text{(parallel z—axis)}$$

Maximum bending stress at A:



$$\sigma_b = \frac{1}{I_z}\, |M_z|\, \frac{d_0}{2} = \frac{m\, r\, \omega^2\, R \left(\frac{d_0}{2}\right)}{\frac{\pi}{64}\,(d_o^4 - d_i^4)}$$
$$= \frac{(0.1)\,(0.1)\,(31.416^2)\,(0.5)\,(\frac{0.1}{2})}{\frac{\pi}{64}\,(0.1^4 - 0.08^4)} = 8.5124\,(10^4)\ \text{N/m}^2$$

Maximum torsional stress at A:

$$\sigma_t = \frac{1}{J_y}\, |M_y| \left(\frac{d_o}{2}\right) = \frac{m\, r\, \omega^2\, R \left(\frac{d_o}{2}\right)}{\frac{\pi}{32}\,(d_o^4 - d_i^4)}$$
$$= 4.2562\,(10^4)\ \text{N/m}^2$$

3.18 Total stiffness with steel specimen:

$$k_{eq} = k_1 + k_2 = 10{,}217.0296 + 750{,}000.0 = 760{,}217.0296\ \text{lb/in}$$

Force in specimen due to magnets (static) due to elongation $X = k_2\, X$.
Force in specimen due to a.c. current in magnets (dynamic) due to elongation $X = k_{eq}\, X - m\, \omega^2\, X$.

$$\text{Ratio of stresses} = \left| \frac{k_2\, X}{k_{eq}\, X - m\, \omega^2\, X} \right| = \frac{1}{2} \quad \text{i.e.,} \quad \left| \frac{k_2}{k_{eq} - m\, \omega^2} \right| = \frac{1}{2}.\text{sp}$$

$$\text{i.e.,} \quad \left| \frac{750{,}000.0}{760{,}217.0296 - \left(\frac{40}{386.4}\right) \omega^2} \right| = \frac{1}{2}$$

Squaring both sides of this equation and rearranging gives:

$$107.1225\, \omega^4 - 15.7365\,(10^8)\, \omega^2 - 167.207\,(10^{14}) = 0$$
$$\text{or} \quad \omega^2 = 0.218378\,(10^8) \quad \text{(positive value)}$$
$$\omega = 4673.0935\ \text{rad/sec} = 743.7442\ \text{Hz}$$

3.19 Equation of motion: $m_{eq}\, \ddot{x} + k_{eq}\, x = F(t)$
where m_{eq} = mass of valve and valve rod plus mass of spring at end = $(20 + (15/3))/386.4 = 0.0647$ lb—sec²/in.
$k_{eq} = 400$ lb/in, $F(t) = A\ p(t) = A\ p_0 \sin \omega t = 100\,(10) \sin \omega t = 1000 \sin 8t$ lb.
Response of valve (steady state) $= x_p(t) = X \sin 8t$ in where

$$X = \frac{1000}{400 - 0.0647\,(8)^2} = 2.5261\ \text{in}$$

3.20 (a) Equation of motion:

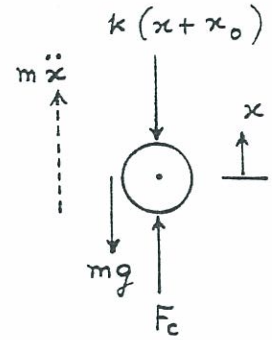
$$m_0\, \ddot{x} + k\,(x + x_0) + m_0\, g = F_c;\ \ \ddot{x} > 0 \tag{1}$$

where F_c = force exerted on the follower by the cam, m_0 = mass of follower plus one third the mass of the spring, and x_0 = initial displacement of the spring.

(b) Force exerted on the follower by the cam:

$$F_c = m_0\, \ddot{x} + k\,(x + x_0) + m_0\, g \tag{2}$$

with $x = e \cos \omega t$.



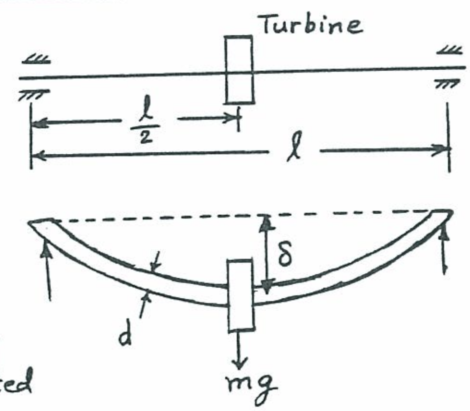
(c) Condition under which follower loses contact with the cam is when F_c is zero and $\ddot{x}$ is negative. Equation (1) can be used to state this condition as:

$$k\,(x + x_0) + m_0\, g \geq |\, m_0\, \ddot{x}\,| \tag{3}$$

3.21 δ_{st} = static radial displacement of shaft under weight of turbine

δ = radial deflection of shaft during rotation

$k = \dfrac{48\,E\,I}{l^3}$ = stiffness of centrally loaded simply supported beam



$$m\,\delta\,\omega^2 = k\,(\delta - \delta_{st}) \quad \text{or} \quad m\,\omega^2 = k - k\cdot\frac{\delta_{st}}{\delta}$$

$$\text{or} \quad \frac{\delta}{\delta_{st}} = \frac{k}{k - m\,\omega^2} \tag{E$_1$}$$

Critical speed is $\quad \omega_{cri} = \sqrt{\dfrac{k}{m}}$ (E$_2$)

If critical speed $= \frac{1}{5}$th of operating speed,

$$\sqrt{\frac{k}{m}} = \frac{1}{5}\,\omega \tag{E$_3$}$$

Here $m = 500/386.4 = 1.2940$ lb-s^2/in

and $\omega = 3000 \times 2\pi/60 = 314.16$ rad/sec

For solid shaft (steel) of diameter d and length l, Eq. (E$_3$) gives

$$\frac{48\,E\,I}{m\,l^3} = \frac{\omega^2}{25} \quad \text{with } E = 30 \times 10^6 \text{ psi and } I = \frac{\pi d^4}{64}$$

i.e., $$\frac{l^3}{d^4} = 13836.8 \qquad (E_4)$$

Let $l = 30\,d$ in Eq. (E_4):

$$d = \frac{27000}{13836.8} = 1.9513 \text{ inch} \quad \text{and hence } l = 58.5395 \text{ inch.}$$

3.22

$$I = \frac{\pi}{64}\left(d_o^4 - d_i^4\right) = \frac{\pi}{64}\left(4^4 - 3.5^4\right) = 5.2002 \text{ in}^4$$

$$k = \frac{48\,E\,I}{l^3} = \frac{48\,(30 \times 10^6)\,(5.2002)}{(100)^3} = 7488.288 \text{ lb/in}$$

$$m = 500/386.4 = 1.2940 \text{ lb-s}^2/\text{in}$$

$$\omega_n = \sqrt{k/m} = \sqrt{7488.288/1.2940} = 76.0719 \text{ rad/sec}$$

Eccentricity $= r = 2$ in, eccentric mass $= m_o = \frac{0.5}{386.4}\ \frac{\text{lb-s}^2}{\text{in}}$

Radial force due to eccentric mass at resonance

$$= F_0 = m_o\, r\, \omega^2 = \left(\frac{5}{386.4}\right)(2)(76.0719)^2 = 149.7654 \text{ lb}$$

Let $x(t)$ = radial displacement of turbine.

At resonance, Eq. (3.15) gives, for $x_o = \dot{x}_o = 0$,

$$x(t) = \frac{1}{2}\,\delta_{st}\,\omega_n t\, \sin \omega_n t$$

where $$\delta_{st} = \frac{F_0}{k} = \frac{149.7654}{7488.288} = 0.02 \text{ in}$$

To activate the limit switch, $x(t) = 0.5$ in. and hence

$$0.5 = \frac{1}{2}(0.02)(76.0719)\,t\, \sin 76.0719\,t$$

i.e., $$t \sin 76.0719\,t = 0.6573 \qquad (E_1)$$

Eq. (E_1) is solved by trial and error (assuming values of $t = 1.0, 0.9, 0.8, 0.7$, etc.) as

$$t \simeq 0.6760 \text{ sec.}$$

3.23

Tip load $= 0.1$ lb, tip mass $= m_o = \frac{0.1}{386.4} = 2.588 \times 10^{-4}$ lb-s^2/in

$$I = \frac{1}{12}(0.2)(0.05)^3 = 2.0833 \times 10^{-6} \text{ in}^4$$

$$k = \frac{3\,E\,I}{l^3} = \frac{3\,(30 \times 10^6)\,(2.0833 \times 10^{-6})}{(10)^3} = 0.1875 \text{ lb/in}$$

$$m = \text{mass of beam} = \frac{0.283}{386.4}\,(10 \times 0.2 \times 0.05) = 7.324 \times 10^{-5} \text{ lb-s}^2\text{/in}$$

$$\omega_n = \left(\frac{k}{m_0 + 0.23\ m}\right)^{\frac{1}{2}} = \left[\frac{0.1875}{(2.588 + 0.7324)\ 10^{-4}}\right]^{\frac{1}{2}} = 5.6469 \ \frac{\text{rad}}{\text{sec}}$$

Eq. (3.68) gives

$$\frac{X}{Y} = \left\{\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}\right\}^{\frac{1}{2}}$$

i.e.,

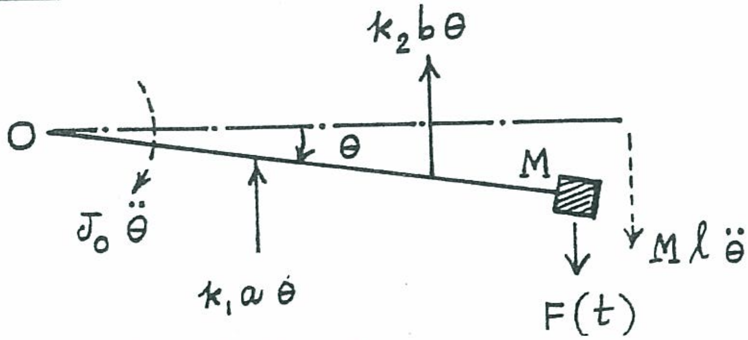
$$\frac{2.5}{0.05} = 50 = \left\{\frac{1 + (2 \times 0.01\ r)^2}{(1-r^2)^2 + (2 \times 0.01\ r)^2}\right\}^{\frac{1}{2}}$$

i.e. $\quad r^4 - 1.9996\ r^2 + 0.9996 = 0$

i.e. $\quad r = \dfrac{\omega}{\omega_n} = 0.9999$

$\therefore \quad \omega = 0.9999\ \omega_n = 5.6463$ rad/sec.

3.24



Equation of motion for rotational motion about the hinge O:

$$(J_0 + M\,\ell^2)\,\ddot{\theta} + (k_1\ a^2 + k_2\ b^2)\,\theta = F(t)\ \ell = F_0\ \ell\ \sin \omega\, t \tag{1}$$

Steady state response (using Eqs. (3.3) and (3.6)):

$$\theta_p(t) = \Theta \sin \omega\, t \tag{2}$$

$$\text{where} \quad \Theta = \frac{F_0\ \ell}{(k_1\ a^2 + k_2\ b^2) - (J_0 + M\,\ell^2)\,\omega^2} \tag{3}$$

$$\text{and} \quad J_0 = \frac{m\,\ell^2}{12} + m\,(\frac{\ell}{2})^2 = \frac{1}{3}\ m\,\ell^2 \tag{4}$$

For given data, $J_0 = \dfrac{1}{3}\,(10)\,(1^2) = 3.3333$ kg–m^2, $\omega = \dfrac{1000\ (2\ \pi)}{60} = 104.72$ rad/sec,

and

$$\Theta = \frac{500\ (1)}{5000\ (0.25^2\ + 0.5^2) - (3.3333 + 50\ (1^2))\ (104.72^2)} = -\ 8.5718\ (10^{-4})\ \text{rad}$$

3.25 Equation of motion for rotation about O:

$$J_0\ \ddot{\theta} = -\ k\ \frac{\theta\,\ell}{4}\ \frac{\ell}{4} - k\ \frac{\theta\,3\,\ell}{4}\ \frac{3\,\ell}{4} + M_0\ \cos\omega\,t$$

$$\text{i.e.,}\quad J_0\ \ddot{\theta} + \left(\frac{5}{8}\ k\ \ell^2\right)\ \theta = M_0\ \cos\omega\,t$$

where $J_0 = \frac{1}{12}\ m\ \ell^2 + m\ (\frac{\ell}{4})^2 = \frac{7}{48}\ m\ \ell^2 = \frac{7}{48}\ (10)\ (1^2) = 1.4583\ \text{kg–m}^2$

and $\omega = 1000$ rpm $= 104.72$ rad/sec. Steady state solution is:

$$\theta_p(t) = \Theta\ \cos\omega\,t$$

where

$$\Theta = \frac{M_0}{\frac{5}{8}\ k\ \ell^2 - J_0\ \omega^2} = \frac{100}{5000\ (\frac{5}{8})\ (1^2) - 1.4583\ (104.72^2)} = -\ 0.007772\ \text{rad}$$

3.26 $k = 4000$ N/m, $m = 10$ kg, $c = 40$ N-s/m, $F(t) = 200 \cos 10t$,

$F_0 = 200$ N, $\omega = 10$ rad/s, $x_0 = 0.1$ m, $\dot{x}_0 = 0$

$\omega_n = \sqrt{\frac{k}{m}} = 20$ rad/s, $\delta_{st} = \frac{F_0}{k} = \frac{200}{4000} = 0.05$ m

$\zeta = c/c_c = (c/2\sqrt{km}) = (40/2\sqrt{4000(10)}) = 0.1$

$\omega_d = \sqrt{1-\zeta^2}\ \omega_n = \sqrt{1-(0.1)^2}\ (20) = 19.899749$ rad/s

$r = \frac{\omega}{\omega_n} = \frac{10}{20} = 0.5$

$$X = \delta_{st}\Big/\sqrt{(1-r^2)^2 + (2\zeta r)^2} = \frac{0.05}{\left\{(1-0.5^2)^2 + (2\,(0.1)(0.5))^2\right\}^{\frac{1}{2}}}$$

$= 0.066082$ m

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}\left(\frac{2 * 0.1 * 0.5}{1-0.5^2}\right) = 0.132552 \text{ rad}$$

steady state response, Eq. (3.25):

$x_p(t) = X \cos(\omega t - \phi)$

$= 0.066082 \cos(10t - 0.132552)$ m

Total response, Eq. (3.35):

$$x(t) = X_0 e^{-\zeta \omega_n t} \cos(\omega_d t - \phi_0) + X \cos(\omega t - \phi) \qquad (E.1)$$

Using the initial conditions x_0 and $\dot{x}_0$, Eq. (E.1) gives

$$x_0 = X_0 \cos \phi_0 + X \cos \phi \qquad (E.2)$$

or $$X_0 \cos \phi_0 = x_0 - X \cos \phi \qquad (E.3)$$

$$\dot{x}_0 = -\zeta \omega_n X_0 \cos \phi_0 + \omega_d X_0 \sin \phi_0 + \omega X \sin \phi \qquad (E.4)$$

or $$X_0 \sin \phi_0 = \frac{1}{\omega_d} \left\{ \dot{x}_0 + \zeta \omega_n X_0 \cos \phi_0 - \omega X \sin \phi \right\} \qquad (E.5)$$

For known values, Eqs. (E.3) and (E.5) yield

$$X_0 \cos \phi_0 = 0.034498, \qquad X_0 \sin \phi_0 = -0.000922$$

Hence

$$X_0 = \left\{ (X_0 \cos \phi_0)^2 + (X_0 \sin \phi_0)^2 \right\}^{\frac{1}{2}} = 0.034510$$

$$\phi_0 = \tan^{-1} \left(\frac{X_0 \sin \phi_0}{X_0 \cos \phi_0} \right) = -0.026710$$

Thus the total response, Eq. (E.1), will be

$$x(t) = 0.034510 \, e^{-2t} \cos(19.899749\, t + 0.026710)$$
$$+ 0.066082 \cos(10t - 0.132552) \text{ m} \qquad (E.6)$$

3.27 $k = 4000$ N/m, $m = 10$ kg, $c = 40$ N-s/m, $F(t) = 200 \cos 10t$,
$F_0 = 200$ N, $\omega = 10$ rad/s, $x_0 = 0$, $\dot{x}_0 = 10$ m/s

From solution of Problem 3.26,

$\zeta = 0.1$, $\omega_d = 19.899749$ rad/s, $r = 0.5$, $X = 0.066082$ m,
$\phi = 0.132552$ rad

$$x_p(t) = 0.066082 \cos(10t - 0.132552) \text{ m}$$

$$X_0 \cos \phi_0 = x_0 - X \cos \phi = -0.065502$$

$$X_0 \sin \phi_0 = \frac{1}{\omega_d} \left\{ \dot{x}_0 + \zeta \omega_n X_0 \cos \phi_0 - \omega X \sin \phi \right\} = 0.491547$$

$$X_0 = \left\{ (X_0 \cos \phi_0)^2 + (X_0 \sin \phi_0)^2 \right\}^{\frac{1}{2}} = 0.495892$$

$$\phi_0 = \tan^{-1} \left(\frac{X_0 \sin \phi_0}{X_0 \cos \phi_0} \right) = -1.438320$$

Thus the total response, Eq. (3.35), is given by

$$x(t) = 0.495892\ e^{-2t} \cos(19.899749\,t + 1.438320)$$
$$+ 0.066082 \cos(10\,t - 0.132552) \quad m$$

3.28 $k = 4000$ N/m, $m = 10$ kg, $c = 40$ N-s/m, $F(t) = 200 \cos 20t$

$F_0 = 200$ N, $\omega = 20$ rad/s, $x_0 = 0.1$ m, $\dot{x}_0 = 0$

$$\omega_n = \sqrt{\frac{k}{m}} = 20 \text{ rad/s}, \quad \delta_{st} = \frac{F_0}{k} = \frac{200}{4000} = 0.05$$

$$\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = \frac{40}{2\sqrt{4000\,(10)}} = 0.1$$

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n = \sqrt{1-0.1^2}\,(20) = 19.899749 \text{ rad/s}$$

$$r = \frac{\omega}{\omega_n} = 1$$

$$X = \frac{\delta_{st}}{\left\{(1-r^2)^2 + (2\zeta r)^2\right\}^{1/2}} = \frac{0.05}{\left\{(1-1^2)^2 + (2*0.1*1)^2\right\}^{1/2}} = 0.25$$

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$$

steady state response, Eq. (3.25):

$$x_p(t) = X \cos(\omega t - \phi) = 0.25 \cos\left(20t - \frac{\pi}{2}\right) \quad m$$

Total response, Eq. (3.35):

$$x(t) = X_0\, e^{-\zeta\omega_n t} \cos(\omega_d t - \phi_0) + X\cos(\omega t - \phi) \qquad (E.1)$$

Using the initial conditions x_0 and $\dot{x}_0$, Eq. (E.1) gives

$$X_0 \cos\phi_0 = x_0 - X\cos\phi = 0.1 - 0.25\cos\frac{\pi}{2} = 0.1$$

$$X_0 \sin\phi_0 = \frac{1}{\omega_d}\left\{\dot{x}_0 + \zeta\omega_n X_0\cos\phi_0 - \omega X \sin\phi\right\}$$

$$= \left(0 + 0.1*20*0.1 - 20*0.25*\sin\frac{\pi}{2}\right)\Big/19.899749$$

$$= -0.241209$$

Hence $X_0 = \left\{(X_0\cos\phi_0)^2 + (X_0 \sin\phi_0)^2\right\}^{\frac{1}{2}} = 0.261117$

$$\phi_0 = \tan^{-1}\left(\frac{X_0 \sin\phi_0}{X_0\cos\phi_0}\right) = \tan^{-1}\left(\frac{-0.241209}{0.1}\right) = -1.177783$$

Total response:

$$x(t) = 0.261117\ e^{-2t} \cos(19.899749\ t + 1.777828)$$
$$+ 0.25 \cos\left(20t - \frac{\pi}{2}\right) \text{ m}$$

3.29 $k = 4000$ N/m, $m = 10$ kg, $c = 40$ N-s/m,

$F(t) = 200 \cos 20t$ N, $F_0 = 200$ N, $\omega = 20$ rad/s

$x_0 = 0$, $\dot{x}_0 = 10$ m/s

From solution of Problem 3.28,

$\zeta = 0.1$, $\omega_n = 20$ rad/s, $\omega_d = 19.899749$ rad/s, $r = 1$

$X = 0.25$, $\phi = \frac{\pi}{2}$

$x_p(t) = 0.25 \cos\left(20t - \frac{\pi}{2}\right)$ m

$X_0 \cos\phi_0 = x_0 - X\cos\phi = 0 - 0 = 0$

$$X_0 \sin\phi_0 = \frac{1}{\omega_d}\left\{\dot{x}_0 + \zeta\omega_n X_0 \cos\phi_0 - \omega X \sin\phi\right\}$$
$$= \frac{1}{19.899749}\left\{10 + 0.1 * 20 * 0 - 20 * 0.25 * \sin\frac{\pi}{2}\right\}$$
$$= 0.251260$$

Hence $X_0 = \left\{(X_0\cos\phi_0)^2 + (X_0\sin\phi_0)^2\right\}^{\frac{1}{2}} = 0.251260$

$$\phi_0 = \tan^{-1}\left(\frac{X_0 \sin\phi_0}{X_0 \cos\phi_0}\right) = 1.570793$$

Total response:

$$x(t) = 0.251260\ e^{-2t} \cos(19.899749\ t - 1.570793)$$
$$+ 0.25 \cos\left(20t - \frac{\pi}{2}\right) \text{ m}$$

3.30 $m = \dfrac{500}{386.4}$ lb—sec^2/in, $F(t) = 200 \sin 100\,\pi\, t$ lb. Let $X_{max} = 0.05$ in < 0.1 in (maximum permissible value). From Eq. (3.33),

$$X_{max} = \delta_{st} \frac{1}{2\,\varsigma \sqrt{1-\varsigma^2}} = 0.05 \tag{1}$$

Let $\varsigma = 0.01$. Then $\delta_{st} = \dfrac{F_0}{k_{eq}} = \dfrac{200}{k_{eq}}$ and Eq. (1) gives

$$k_{eq} = \frac{200}{2\,(0.01)\sqrt{1-0.0001}\;(0.05)} = 20.0020\,(10^4)\ \text{lb/in}$$

Since shock mounts are in parallel, stiffness of each mount $= k = \dfrac{k_{eq}}{3} = 6.6673\,(10^4)$ lb/in.

$$\varsigma = \frac{c_{eq}}{c_c} = \frac{c_{eq}}{\sqrt{2\,k_{eq}\,m}}$$

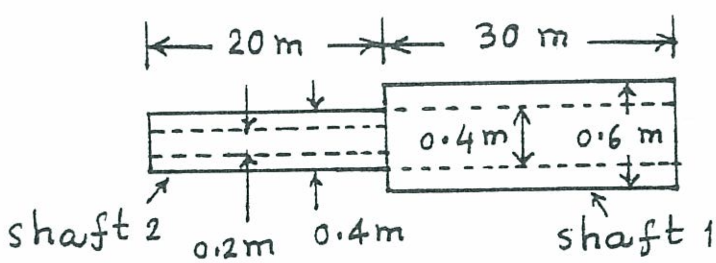
or $\quad c_{eq} = \varsigma\sqrt{2\,k_{eq}\,m} = 0.01\sqrt{2\,(20.0020\,(10^4))\,(\dfrac{500}{386.4})} = 7.1948$ lb—sec/in

and hence $c = \dfrac{c_{eq}}{3} = 2.3983$ lb-sec/in

3.31 Equation of motion for torsional system:

$$J_0\,\ddot{\theta} + c_t\,(\dot{\theta} - \dot{\alpha}) + k_t\,(\theta - \alpha) = 0 \tag{1}$$

where $\theta =$ angular displacement of shaft and $\alpha =$ angular displacement of base of shaft $= \alpha_0 \sin \omega\, t$. Steady state response of propeller (Eq. (3.67)):



$$\theta_p(t) = \Theta \sin(\omega t - \phi) \tag{2}$$

$$\text{where } \Theta = \alpha_0 \left\{ \frac{k_t^2 + (c_t\ \omega)^2}{(k_t - J_0\ \omega^2)^2 - (c_t\ \omega)^2} \right\}^{\frac{1}{2}} \tag{3}$$

$$\text{and } \phi = \tan^{-1} \left\{ \frac{J_0\ c_t\ \omega^3}{k_t\ (k_t - J_0\ \omega^2) + (c_t\ \omega)^2} \right\} \tag{4}$$

Here $J_0 = 10^4$ kg–m^2, $\varsigma_t = 0.1$, and $\omega = 314.16$ rad/sec. Torsional stiffnesses of shafts:

$$(k_t)_1 = \frac{G_1\ J_1}{\ell_1} = \frac{(80\ (10^9)) \left(\frac{\pi}{32}\ (0.6^4 - 0.4^4)\right)}{30} = 27.2272\ (10^6)\ \text{N–m/rad}$$

$$(k_t)_2 = \frac{G_2\ J_2}{\ell_2} = \frac{(80\ (10^9)) \left(\frac{\pi}{32}\ (0.4^4 - 0.2^4)\right)}{20} = 9.4248\ (10^6)\ \text{N–m/rad}$$

Series springs give:

$$k_t = \frac{(k_t)_1\ (k_t)_2}{(k_t)_1 + (k_t)_2} = \frac{(27.2272\ (10^6))\ (9.4248\ (10^6))}{27.2272\ (10^6) + 9.4248\ (10^6)} = 7.0013\ (10^6)\ \text{N–m/rad}$$

$$c_t = \varsigma\ (2\sqrt{J_0\ k_t}) = 0.1\ (2)\ \sqrt{(10^4)\ (7.0013\ (10^6))} = 52{,}919.8624\ \text{N–m–s/rad}$$

From Eq. (3),

$$\Theta = 0.05 \left[\frac{(7.0013\ (10^6))^2 + \left\{5.2920\ (10^4)\ (314.16^2)\right\}^2}{\left\{7.0013\ (10^6) - (10^4)\ (314.16^2)\right\}^2 + \left\{5.2920\ (10^4)\ (314.16)\right\}^2} \right]^{\frac{1}{2}}$$

$$= 9.2028\ (10^{-4})\ \text{rad}$$

$$\phi = \tan^{-1} \left\{ \frac{(10^4)\ (5.2920\ (10^4))\ (314.16^3)}{7.0013\ (10^6)\ \left[7.0013\ (10^6) - (10^4)\ (314.16^2)\right] + (5.2920\ (10^4)\ (314.16))^2} \right\}$$

$$= \tan^{-1}\ (59.3664) = 89.0350^\circ = 1.5540\ \text{rad}$$

3.32 $$X = \frac{\delta_{st}}{\{(1-r^2)^2 + (2\zeta r)^2\}^{1/2}}$$

For maximum X, $$\frac{dX}{dr} = -\delta_{st} \cdot \frac{1}{2} \frac{1}{\{(1-r^2)^2 + (2\zeta r)^2\}^{3/2}} \cdot \{2(1-r^2)(-2r) + 2(2\zeta r)(2\zeta)\} = 0$$

i.e., $$-4r(1-r^2) + 8r\zeta^2 = 0$$

i.e., $$r = \sqrt{1-2\zeta^2}$$

$$X\Big|_{at\ r=\sqrt{1-2\zeta^2}} = \frac{\delta_{st}}{\left[\{1-(1-2\zeta^2)\}^2 + \left(2\zeta\sqrt{1-2\zeta^2}\right)^2\right]^{1/2}} = \frac{\delta_{st}}{2\zeta\sqrt{1-\zeta^2}}$$

$$\therefore \quad \left(\frac{X}{\delta_{st}}\right)_{max} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

3.33 Under a d.c. current (I) through the coil, core rotates by angle θ. Torque developed due to I balances the restoring torque of spring: $a\ I = k_t\ \theta$ where a is a constant and k_t is the torsional spring constant. Under an a.c. current I(t), torque developed is $T(t) = a\ I(t)$ and the equation of motion is:

$$J_0\ \ddot{\theta} + c_t\ \dot{\theta} + k_t\ \theta = T(t) = a\ I(t) = a\ I_0 \cos \omega t \tag{1}$$

Steady state angular displacement of core:

$$\theta_p(t) = \Theta \cos(\omega t - \phi).sp \tag{2}$$

$$\text{where} \quad \Theta = \frac{a\ I_0}{\left\{(k_t - J_0\ \omega^2)^2 + (c_t\ \omega)^2\right\}^{\frac{1}{2}}} = \frac{\left(\frac{a\ I_0}{k_t}\right)}{\left[\left\{1 - (\frac{\omega}{\omega_n})^2\right\}^2 + \left\{2\ \varsigma\ \frac{\omega}{\omega_n}\right\}^2\right]^{\frac{1}{2}}} \tag{3}$$

When $\omega = 0$ (d.c. current) and $I_0 = 1$ ampere, Eq. (1) gives

$$\Theta_{dc} = \left(\frac{a}{k_t}\right) = 1 \text{ (reading corresponding } \Theta_{dc})$$

and hence $a = k_t = 62.5$.

When $\omega = 50$ Hz $= 314.16$ rad/sec and $I_0 = 5$ amperes, Eq. (3) gives:

$$\Theta_{ac} = \frac{\left(\frac{a\ (5)}{k_t}\right)}{\left[\left\{1 - \left(\frac{314.16}{250}\right)^2\right\}^2 + \left\{2\ (1)\left(\frac{314.16}{250}\right)\right\}^2\right]^{\frac{1}{2}}} = 1.9386 \text{ amperes}$$

where $J_0 = 0.001$ N–m^2, $k_t = 62.5$ N-m/rad, $c_t = 0.5$ N-m-s/rad, and $\omega_n = \sqrt{\frac{k_t}{J_0}} = \sqrt{\frac{62.5}{0.001}} = 250$ rad/s. The steady state value of current indicated by ammeter $= 1.9386$ amperes (this shows that the ammeter is not accurate).

3.34 Eq. (3.34): $\dfrac{X_{res}}{\delta_{st}} = \left.\dfrac{X}{\delta_{st}}\right|_{\omega=\omega_n} = \dfrac{1}{2\zeta}$

i.e., $$\delta_{st} = 2\,\zeta\left(\frac{20}{1000}\right) = 0.04\,\zeta \qquad (E_1)$$

Eq. (3.30): $$\frac{X}{\delta_{st}} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \; ; \quad r = 0.75 = \frac{\omega}{\omega_n}$$

i.e., $$\frac{0.01}{\delta_{st}} = \frac{1}{\sqrt{(1-0.75^2)^2 + (2\,\zeta \times 0.75)^2}} \qquad (E_2)$$

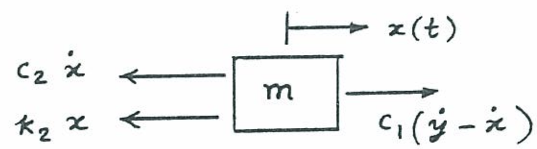
Eqs. (E_1) and (E_2) give

$$\frac{0.01}{0.04\,\zeta} = \frac{1}{\sqrt{0.1914 + 2.25\,\zeta^2}}$$

i.e., $0.1914 + 2.25\,\zeta^2 = 16\,\zeta^2$

i.e., $\zeta = 0.1180$

3.35 (a) Equation of motion of mass:



$$m\ddot{x} = c_1(\dot{y}-\dot{x}) - c_2\dot{x} - k_2 x$$

i.e., $$m\ddot{x} + (c_1+c_2)\,\dot{x} + k_2 x = c_1\,\dot{y} = -c_1\omega Y \sin\omega t$$

(b) $$x_p(t) = \frac{-(c_1\omega Y/k_2)}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \sin(\omega t - \phi)$$

where $r = \omega/\omega_n$, $\zeta = (c_1+c_2)\,\omega/(2 r k)$ and $\phi = \tan^{-1}\left(\dfrac{2\zeta r}{1-r^2}\right)$.

(c) steady-state force transmitted to point P:

$$= k_2 x_p + c_2 \dot{x}_p$$

$$= \frac{-(c_1\omega Y)}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \left\{ \sin(\omega t - \phi) + \frac{c_2\omega}{k_2}\cos(\omega t - \phi) \right\}$$

3.40 Eq. (3.34) gives $\left(\frac{X}{\delta_{st}}\right)_{\omega=\omega_n} = \frac{1}{2\zeta}$

If $X = \frac{1}{\sqrt{2}} X_{max} = \frac{1}{\sqrt{2}} X\Big|_{\omega=\omega_n}$, Eq. (3.30) gives

$$\frac{1}{\sqrt{2}} \cdot \frac{1}{2\zeta} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

Squaring and rearranging

$$8\zeta^2 = (1-r^2)^2 + 4\zeta^2 r^2 = 1 - 2r^2 + r^4 + 4r^2\zeta^2$$

$$r^4 + r^2(4\zeta^2 - 2) + (1 - 8\zeta^2) = 0$$

$$r^2 = 1 - 2\zeta^2 \pm 2\zeta\sqrt{1+\zeta^2}$$

Neglecting terms involving ζ^2,

$$r^2 = \frac{\omega^2}{\omega_n^2} = 1 \pm 2\zeta .$$

Let $\omega = \omega_1$ when $r^2 = 1 - 2\zeta$ and $\omega = \omega_2$ when $r^2 = 1 + 2\zeta$

$$\frac{\omega_2^2 - \omega_1^2}{\omega_n^2} = \frac{(\omega_2 + \omega_1)(\omega_2 - \omega_1)}{\left(\frac{\omega_2 + \omega_1}{2}\right)^2} = 4\zeta$$

$$\therefore \quad \frac{\omega_2 - \omega_1}{\omega_2 + \omega_1} = \zeta$$

3.41 $k_t = \frac{\pi G}{32\, l} d^4 = \frac{\pi (79.3 \times 10^9)}{32 (1)} \left(\frac{4}{100}\right)^4 = 19930.31$ N-m/rad

$$\omega_n = \sqrt{\frac{k_t}{J_0}} = \sqrt{\frac{19930.31}{10}} = 44.6434 \text{ rad/sec}$$

$$\theta_{st} = M_{t0}/k_t = 1000/19930.31 = 0.0502 \text{ rad}$$

$$\zeta_t = \frac{c_t}{2 J_0 \omega_n} = \frac{300}{2(10)(44.6434)} = 0.336$$

(a) Eq. (3.30), when written for a torsional system, gives

$$\frac{\Theta}{\theta_{st}} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

i.e., $$\frac{(2/57.2956)}{0.0502} = \frac{1}{\sqrt{(1-r^2)^2 + (2 \times 0.336\, r)^2}}$$

i.e., $$r^4 - 1.5484\, r^2 - 1.0679 = 0$$

i.e., $$r^2 = 2.0655,\ -0.5171$$

$$\therefore \quad \omega = r\,\omega_n = \sqrt{2.0655}\,(44.6434) = 64.16 \text{ rad/sec}$$

(b) Maximum torque transmitted to the support:

$$M_t(t) = k_t\, \theta(t) + c_t\, \dot\theta(t)$$

$$= k_t\, \Theta \cos(\omega t - \phi) - c_t\, \Theta\, \omega \sin(\omega t - \phi)$$

$$(M_t)_{max} = \sqrt{(k_t\,\Theta)^2 + (c_t\,\Theta\,\omega)^2}$$

$$= \sqrt{\left\{19930.31\left(\frac{2}{57.2956}\right)\right\}^2 + \left\{300\left(\frac{2}{57.2956}\right)(64.16)\right\}^2}$$

$$= 967.2 \text{ N-m}$$

3.42 Complete solution is $x(t) = X_0\, e^{-\zeta\omega_n t}\cos(\omega_d t + \phi_0) + X\cos(\omega t - \phi)$

$\omega = 2\pi(3.5) = 21.9912$ rad/sec, $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2500}{10}} = 15.8114$ rad/sec

$$\delta_{st} = \frac{F_0}{k} = \frac{180}{2500} = 0.072 \text{ m}$$

$$\zeta = \frac{c}{2m\omega_n} = \frac{45}{2(10)(15.8114)} = 0.1423, \quad r = \frac{\omega}{\omega_n} = \frac{21.9912}{15.8114} = 1.3908$$

$$\zeta\omega_n = 2.25, \quad \omega_d = \sqrt{1-\zeta^2}\,\omega_n = 15.6505$$

$$X = \frac{\delta_{st}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{0.072}{\left[(1 - 1.3908^2)^2 + (2 \times 0.1423 \times 1.3908)^2\right]^{1/2}}$$

$$= 0.07095 \text{ m}$$

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}\left(\frac{0.3958}{-0.9343}\right) = -22.9591^\circ$$

$$x(t) = X_0\, e^{-2.25t}\cos(15.6505t + \phi_0) + 0.07095\cos(21.9912t + 22.9591^\circ)$$

$$\dot x(t) = -2.25\, X_0\, e^{-2.25t}\cos(15.6505t + \phi_0) - 15.6505\, X_0\, e^{-2.25t}\sin(15.6505t + \phi_0)$$
$$- 21.9912\,(0.07095)\sin(21.9912\,t + 22.9591^\circ)$$

$$x(0) = 0.015 = X\cos\phi_0 + 0.07095\cos 22.9591^\circ$$

$$X\cos\phi_0 = -0.05033 \quad ---- \quad (E_1)$$

$$\dot x(0) = 5 = -2.25\, X_0\cos\phi_0 - 15.6505\, X_0\sin\phi_0 - 1.5603\sin 22.9591^\circ$$

$$X\sin\phi_0 = \frac{-0.6086 - 2.25\, X_0\cos\phi_0 - 5}{15.6505} = -0.3511 \quad ---- \quad (E_2)$$

Eqs. (E_1) and (E_2) give

$$X_0 = \left\{(-0.05033)^2 + (-0.3511)^2\right\}^{1/2} = 0.3547$$

$$\phi_0 = \tan^{-1}\left(\frac{0.3511}{0.05033}\right) = \tan^{-1}(6.9760) = 81.8423^\circ$$

3.43 (a) Eq. (3.38) gives $\frac{1}{2\zeta} \approx \left(\frac{X}{\delta_{st}}\right)_{max} = \frac{0.2}{0.1} = 2$

$\therefore \ \zeta = 0.25$

(b) Eqs. (3.42) yield

$$\left(\frac{\omega_1}{\omega_n}\right)^2 \approx 1-2\zeta = 0.5 \ , \quad \omega_1 = \omega_n \sqrt{0.5} = (5 \times 2\pi)\sqrt{0.5} = 22.2145 \text{ rad/sec}$$

$$\left(\frac{\omega_2}{\omega_n}\right)^2 \approx 1+2\zeta = 1.5 \ , \quad \omega_2 = \omega_n \sqrt{1.5} = (5 \times 2\pi)\sqrt{1.5} = 38.4766 \text{ rad/sec}$$

3.44 Amplitude of vibration under base excitation:

$$X = Y \left\{ \frac{\sqrt{k^2 + (c\,\omega)^2}}{\left[\left(k - m\,\omega^2\right)^2 + (c\,\omega)^2\right]^{\frac{1}{2}}} \right\}$$

$$= \frac{(0.2)\sqrt{k^2 + c^2\,(157.08)^2}}{\left[\left\{k - 2000\,(157.08)^2\right\}^2 + c^2\,(157.08)^2\right]^{\frac{1}{2}}} = 0.1 \text{ m} \qquad (1)$$

Let $k = 5\,(10^6)$ N/m. Then Eq. (1) gives:

$$\frac{\sqrt{25\,(10^{12}) + 2.4674\,(10^4)\,c^2}}{\sqrt{1966.7717\,(10^{12}) + 2.4674\,(10^4)\,c^2}} = 0.5$$

i.e., $1.85055\,(10^4)\,c^2 = 466.6929\,(10^{12})$ i.e., $c = 158805.0$ N–s/m

---

3.45

$$\frac{X}{Y} = \left[\frac{k^2 + c^2\,\omega^2}{\left(k - m\,\omega^2\right)^2 + c^2\,\omega^2}\right]^{\frac{1}{2}} .sp$$

$$\text{or} \quad \frac{10^{-6}}{Y} = \left[\frac{(10^6) + (10^3\,(200\,\pi))^2}{\left\{10^6 - \left(\frac{5000}{9.81}\right)(200\,\pi)^2\right\}^2 + \left\{(10^3)\,(200\,\pi)\right\}^2}\right]^{\frac{1}{2}}$$

or $Y = 169.5294\,(10^{-6})$ m

3.46 Equation of motion:

$$I_0\,\ddot{\theta} + \left(k\,\frac{\ell}{4}\,\theta\right)\frac{\ell}{4} + \left(c\,\frac{\ell}{4}\,\dot{\theta}\right)\frac{\ell}{4} + \left(k\,\frac{3\,\ell}{4}\,\theta\right)\frac{3\,\ell}{4} = M_0 \cos\omega t$$

$$\text{or} \quad I_0\,\ddot{\theta} + c\,\frac{\ell^2}{16}\,\dot{\theta} + \frac{5}{8}\,k\,\ell^2\,\theta = M_0 \cos\omega t$$

$$\text{where } I_0 = \frac{m\,\ell^2}{12} + m\left(\frac{\ell}{4}\right)^2 = \frac{7}{48}\,m\,\ell^2 = \frac{7}{48}\,(10)\,(1^2) = 1.4583 \text{ kg—m}^2$$

$$\frac{c\,\ell^2}{16} = \frac{(1000)\,(1^2)}{16} = 62.5 \text{ N—m—s/rad}$$

$$\frac{5}{8}\,k\,\ell^2 = \frac{5}{8}\,(5000)\,(1^2) = 3125.0 \text{ N—m/rad}$$

$$\omega = \frac{1000\,(2\,\pi)}{60} = 104.72 \text{ rad/sec}$$

Equation of motion becomes:

$$1.4583\,\ddot{\theta} + 62.5\,\dot{\theta} + 3125.0\,\theta = 100\cos 104.72\,t$$

Steady state response is given by Eq. (3.28):

$$\theta_p(t) = \Theta\cos(\omega\,t - \phi) = \Theta\cos(104.72\,t - \phi).\text{sp}$$

$$\text{where}\quad \Theta = \frac{100}{\left[\left\{3125.0 - 1.4583\,(104.72^2)\right\}^2 + \left\{62.5\,(104.72)\right\}^2\right]^{\frac{1}{2}}} = 0.006927 \text{ rad}$$

$$\text{and}\quad \phi = \tan^{-1}\left(\frac{62.5\,(104.72)}{3125.0 - 1.4583\,(104.72^2)}\right) = -\,0.4705 \text{ rad} = -\,26.9606^\circ$$

3.47 $m = 100$ kg, $F_0 = 100$ N, $X_{max} = 0.005$ m at $\omega = 300$ rpm $= 31.416$ rad/sec.
Equations (3.33) and (3.34) yield:

$$\omega = \omega_n\sqrt{1 - 2\,\zeta^2} = \sqrt{\frac{k}{m}}\,\sqrt{1 - 2\,\zeta^2} = 31.416$$

$$\text{or}\quad k\,(1 - 2\,\zeta^2) = (100)\,(31.416^2) = 98{,}696.5056 \tag{1}$$

$$\text{and}\quad X_{max} = \delta_{st}\,\frac{1}{2\,\zeta\sqrt{1-\zeta^2}} = \frac{F_0}{k}\,\frac{1}{2\,\zeta\sqrt{1-\zeta^2}} = 0.005$$

$$\text{or}\quad k\,\zeta\sqrt{1-\zeta^2} = \frac{F_0}{2\,(0.005)} = 10{,}000.0 \tag{2}$$

Divide Eq. (1) by (2):

$$\frac{1 - 2\,\zeta^2}{\zeta\sqrt{1-\zeta^2}} = 9.8696 \tag{3}$$

Squaring Eq. (3) and rearranging leads to:

$$101.4090\,\zeta^4 - 101.4090\,\zeta^2 + 1 = 0 \quad \text{or} \quad \zeta = 0.0998,\ 0.9950$$

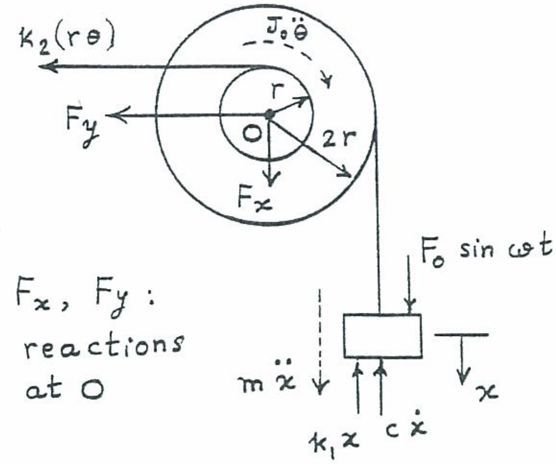
Using $\zeta = 0.0998$ in Eq. (1), we obtain

$$k = \frac{98696.5056}{1 - 2\,(0.0998^2)} = 100{,}702.4994 \text{ N/m}$$

Since $\zeta = \dfrac{c}{2\,m\,\omega_n}$, we find

$$c = 2\,m\,\omega_n\,\zeta = 2\,(100)\sqrt{\frac{100702.4944}{1000}}\,(0.0998) = 633.4038 \text{ N—s/m}$$

3.48



Equation of motion for rotation of pulley about O:

$$-k_2\,(\theta\, r)\, r - J_0\,\ddot{\theta} - k_1\, x\,(2\, r) - c\,\dot{x}\,(2\, r) + F_0 \sin \omega\, t\,(2\, r) - m\,\ddot{x}\,(2\, r) = 0 \qquad (1)$$

where $\theta = x/(2r)$. Equation (1) can be rearranged as:

$$\left(\frac{J_0}{2\, r} + 2\, m\, r\right)\ddot{x} + 2\, c\, r\, \dot{x} + \left(2\, k_1\, r + \frac{1}{2}\, k_2\, r\right) x = 2\, r\, F_0 \sin \omega\, t \qquad (2)$$

For given data, Eq. (2) becomes

$$11\,\ddot{x} + 50\,\dot{x} + 112.5\, x = 5 \sin 20\, t \qquad (3)$$

Steady state response is given by Eq. (3.25):

$$x_p(t) = X \cos\,(\omega\, t - \phi)$$

where $X = \dfrac{5}{\left[\left\{112.5 - 11\,(20^2)\right\}^2 + \left\{50\,(20)\right\}^2\right]^{\frac{1}{2}}} = 0.001136$ m

and $\phi = \tan^{-1}\left(\dfrac{50\,(20)}{112.5 - 11\,(20^2)}\right) = -\,0.2291$ rad $= -\,13.1287°$

3.49 (a)

$$\sum M_0 = 0 \text{ (about hinge):}$$

$$I_0\,\ddot{\theta} + \left(k\,\theta\,\frac{3\,\ell}{4}\right)\frac{3\,\ell}{4} + (c\,\ell\,\dot{\theta})\,\ell = \frac{\ell}{2}\, F_0 \sin \omega\, t$$

$$\text{or}\quad I_0\,\ddot{\theta} + c\,\ell^2\,\dot{\theta} + \frac{9}{16}\, k\,\ell^2\,\theta = \frac{F_0\,\ell}{2}\sin \omega\, t$$

Magnitude of steady state response:

$$\Theta_a = \left(\frac{F_0\,\ell}{2}\right) \Big/ \left[\left\{\frac{9}{16}\, k\,\ell^2 - I_0\,\omega^2\right\}^2 + (c\,\ell^2\,\omega)^2\right]^{\frac{1}{2}} \qquad (1)$$

(b)

$$\sum M_0 = 0 \ \text{(about hinge):}$$

$$I_0\, \ddot{\theta} + (k\, \ell\, \theta)\, \ell + \left(c\, \frac{3\,\ell}{4}\, \dot{\theta}\right) \frac{3\,\ell}{4} = \frac{\ell}{2}\, F_0 \sin \omega\, t$$

$$\text{or} \quad I_0\, \ddot{\theta} + \frac{9}{16}\, c\, \ell^2\, \dot{\theta} + k\, \ell^2\, \theta = \frac{F_0\, \ell}{2} \sin \omega\, t$$

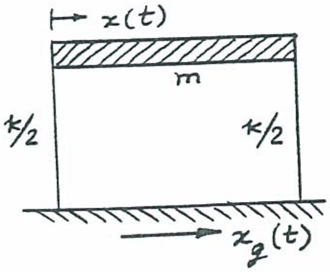
Magnitude of steady state response:

$$\Theta_b = \left(\frac{F_0\, \ell}{2}\right) \Big/ \left[\left\{k\, \ell^2 - I_0\, \omega^2\right\}^2 + \left\{\frac{9}{16}\, c\, \ell^2\, \omega\right\}^2\right]^{\frac{1}{2}} \qquad (2)$$

Usually, c is small compared to k. If the term containing c is negligible, Θ_a will be smaller than Θ_b. Hence arrangement (a) is desirable.

3.52 $\ddot{y}(t) = \ddot{x}_g(t) = A \cos \omega t$; $\dot{y}(t) = \frac{A}{\omega} \sin \omega t + B_1$

$y(t) = -\frac{A}{\omega^2} \cos \omega t + B_1 t + B_2$



Assuming $y(0) = \dot{y}(0) = 0$, we get

$y(t) = -\frac{A}{\omega^2} \cos \omega t$

Equation of motion:

$m\ddot{x} + k(x-y) = 0$

i.e., $m\ddot{z} + kz = -m\ddot{y} = -m\ddot{x}_g(t) = -mA \cos \omega t$

where $z = x - y$

Solution is: $z(t) = \dfrac{-mA \cos \omega t}{k - m\omega^2}$

$\therefore \quad x(t) = z(t) + y(t) = -\left(\dfrac{m}{k - m\omega^2} + \dfrac{1}{\omega^2}\right) A \cos \omega t$

3.53 From solution of problem 3.52,

$$x(t) = \left|\frac{-mA}{k - m\omega^2}\right| \sin \omega t - \frac{A}{\omega^2} \sin \omega t$$

$$= \left|\frac{-2000\left(\frac{100}{1000}\right)}{0.1 * 10^6 - 2000(25)^2}\right| \sin 25t - \left(\frac{100}{1000}\right)\frac{1}{(25)^2} \sin 25t$$

For maximum $x(t)$,

$$x(t) = \left(\frac{-200}{1.15 \times 10^6} - \frac{1}{6250}\right) \sin 25t = -3.3391 \times 10^{-4} \sin 25t \ \text{m}$$

$\therefore$ Maximum horizontal displacement of floor $= 0.3339$ mm

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3.54

$$m(\ddot{x}-\ddot{y}) + k(x-y) = -m\ddot{y} = -m\ddot{x}_g \qquad (E_1)$$

Here $y(t) = x_g(t) = X_g \cos\omega t$, and Eq. (E_1) becomes

$$m\ddot{z} + kz = m\omega^2 X_g \cos\omega t \qquad \text{with } z = x-y$$

Solution is:

$$z(t) = \frac{m\omega^2 X_g \cos\omega t}{k - m\omega^2} = \frac{X_g r^2 \cos\omega t}{1-r^2}$$

with $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{0.5\times10^6}{2000}} = 15.8114$ rad/sec

and $r = \omega/\omega_n = 200/15.8114 = 12.6491$

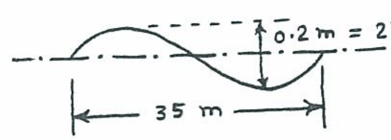
$$z(t) = \left(\frac{15}{1000}\right)\left\{\frac{12.6491^2}{1-12.6491^2}\right\}\cos 200t = -0.01509 \cos 200t \text{ m}$$

$$x(t) = y(t) + z(t) = \{0.015 \cos 200t - |0.01509| \cos 200t\} \text{ m}$$

$\therefore$ Amplitude of vibration of floor $= 0.03009$ m $= 30.09$ mm.

3.55

Time taken by car to travel one cycle (35 m) is



$$\tau = \frac{35 \times 3600}{60\times 1000} = 2.1 \text{ sec}$$

Excitation frequency $= \omega = \frac{2\pi}{\tau} = 2.992$ rad/sec

$\omega_n = 2\pi(2) = 12.5664$ rad/sec, $\quad r = \frac{\omega}{\omega_n} = 0.2381$, $\quad \zeta = 0.15$

Amplitude of vibration of car is given by Eq. (3.68):

$$\frac{X}{Y} = \left[\frac{1+(2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}\right]^{1/2} \qquad (E_1)$$

$$X = 0.1\left\{\frac{1+(2\times0.15\times0.2381)^2}{(1-0.2381^2)^2+(2\times0.15\times0.2381)^2}\right\}^{1/2}$$

$$= 0.105977 \text{ m}$$

The most unfavorable speed corresponds to the maximum of $\frac{X}{Y}$ in Eq. (E_1). For maximum of $\frac{X}{Y}$ with respect to r,

$$\frac{d}{dr}\left[\frac{1+4\zeta^2r^2}{1+r^4-2r^2+4\zeta^2r^2}\right] = 0$$

i.e., $$\frac{(1+r^4-2r^2+4\zeta^2r^2)(8\zeta^2 r) - (1+4\zeta^2r^2)(4r^3-4r+8\zeta^2r)}{(1+r^4-2r^2+4\zeta^2r^2)^2} = 0$$

i.e., $$-4r(2\zeta^2r^4 + r^2 - 1) = 0$$

i.e., $$r = 0 \quad \text{or} \quad r^2 = \frac{-1\pm\sqrt{1+8\zeta^2}}{4\zeta^2}$$

Feasible value of $r^2 = \dfrac{-1 + \sqrt{1 + 8\,(0.15)^2}}{4\,(0.15)^2} = 0.9586$

$$r = \frac{\omega}{\omega_n} = 0.9791$$

$$\omega = 0.9791\,(12.5664) = 12.3035 \text{ rad/sec} = \frac{2\pi}{\tau}$$

where $\tau = \dfrac{35 \times 3600}{s \times 1000}$ and s = speed of car in km/hr.

$$\therefore \quad s = \frac{12.3035 \times 35 \times 3.6}{2\pi} = 246.7279 \text{ km/hr.}$$

3.56 Equations (3.73) and (3.68) give

$$F_T = m\,\omega^2 X = m\,\omega^2\, Y \left[\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}\right]^{1/2}$$

$$\frac{F_T}{kY} = \frac{m\,\omega^2}{k}\left[\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}\right]^{1/2}$$

$$= r^2 \left[\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}\right]^{1/2}$$

3.57 Eq. (3.75): $\quad m\ddot{z} + c\dot{z} + kz = -m\ddot{y} = m\,\omega^2 Y \cos\omega t$

steady-state solution is:

$$z(t) = \frac{m\,\omega^2 Y \cos(\omega t - \phi_1)}{(k - m\omega^2)^2 + (c\omega)^2} = Z \cos(\omega t - \phi_1)$$

where $\phi_1 = \tan^{-1}\left(\dfrac{c\omega}{k - m\omega^2}\right)$

Damping force $= c\,\dfrac{dz}{dt} = -c\,\omega\, Z \sin(\omega t - \phi_1)$

Energy absorbed per cycle by the damper (E):

$$E = \int_{t=0}^{2\pi/\omega} c\,\frac{dz}{dt}\cdot dz = \int_0^{2\pi/\omega} \{-c\,\omega\, Z \sin(\omega t - \phi_1)\}\{-\omega\, Z \sin(\omega t - \phi_1)\}\,dt$$

$$= c\,\omega^2 Z^2 \int_0^{2\pi/\omega} \sin^2(\omega t - \phi_1)\,dt = \pi\, c\,\omega\, Z^2$$

Since $Z = m\,\omega^2 Y \big/ \sqrt{(k - m\omega^2)^2 + (c\omega)^2}$,

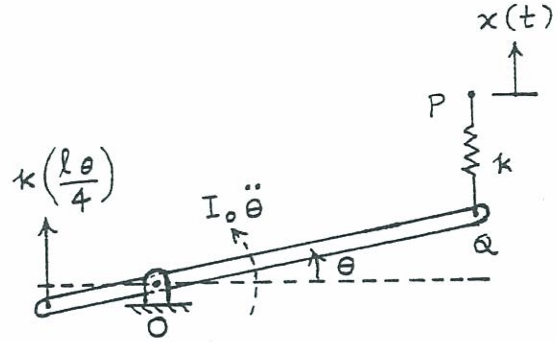
$$E = \left\{\frac{\pi\, c\,\omega\,(m^2\omega^4 Y)}{(k - m\omega^2)^2 + c^2\omega^2}\right\}$$

For maximum power, $\dfrac{dE}{dc} = 0$

i.e., $$\frac{\{(k-m\omega^2)^2+c^2\omega^2\}(\pi\omega^5 m^2 Y)-\pi c\,\omega^5 m^2 Y\,(2c\,\omega^2)}{\{(k-m\omega^2)^2+c^2\omega^2\}^2}=0$$

i.e., $$c=\left(\frac{k-m\omega^2}{\omega}\right).$$

3.58



Linear displacement of point Q due to $\theta = \frac{3\,\ell}{4}\,\theta$ and net compression of spring PQ = $\frac{3}{4}\,\ell\,\theta - x(t)$. Equation of motion:

$$I_0\,\ddot{\theta} = -\,\frac{k\,\ell\,\theta}{4}\,\frac{\ell}{4} - k\left(\frac{3\,\ell\,\theta}{4} - x(t)\right)\frac{3\,\ell}{4} \qquad (1)$$

$$\text{where}\quad I_0 = \frac{1}{12}\,m\,\ell^2 + m\,(\frac{\ell}{4})^2 = \frac{7}{48}\,m\,\ell^2 = \frac{7}{48}\,(10)\,(1^2) = 1.4583\ \text{kg-m}^2$$

Hence Eq. (1) can be rewritten as

$$I_0\,\ddot{\theta} + \left(\frac{5}{8}\,k\,\ell^2\right)\theta = \left(\frac{3}{4}\,k\,\ell\,x_0\right)\sin\omega\,t \qquad (2)$$

Steady state angular displacement of the bar is given by Eq. (3.6):

$$\Theta = \left(\frac{3}{4}\,k\,\ell\,x_0\right)\Big/\left(\frac{5}{8}\,k\,\ell^2 - I_0\,\omega^2\right) \qquad (3)$$

$$= \left(\frac{3}{4}\,(1000)\,(1)\,(0.01)\right)\Big/\left(\frac{5}{8}\,(1000)\,(1^2) - 1.4583\,(10^2)\right) = 0.01565\ \text{rad}$$

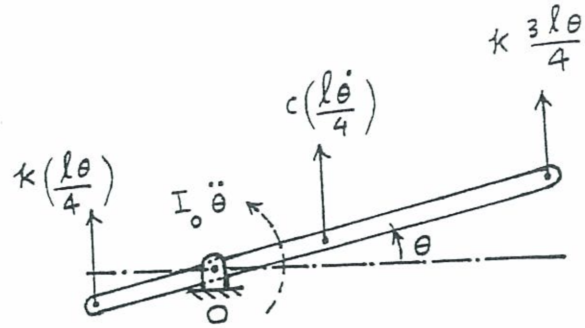
$$\text{and hence}\quad \theta(t) = \Theta\,\sin\omega\,t = 0.01565\,\sin 10\,t\ \text{rad}$$

3.59 Equation of motion:

$$I_0\,\ddot{\theta} = -\,k\,\frac{\ell\,\theta}{4}\,(\frac{\ell}{4}) - c\,\frac{\ell}{4}\,\dot{\theta}\,(\frac{\ell}{4}) - k\left(\frac{3\,\ell}{4}\,\theta - x(t)\right)\frac{3\,\ell}{4}$$

$$\text{i.e.,}\quad I_0\,\ddot{\theta} + \frac{1}{16}\,c\,\ell^2\,\dot{\theta} + \frac{5}{8}\,k\,\ell^2 = \frac{3}{4}\,k\,\ell\,x(t) = \frac{3}{4}\,k\,\ell\,x_0\,\sin\omega\,t \qquad (1)$$

$$\text{where}\quad I_0 = \frac{1}{12}\,m\,\ell^2 + m\left(\frac{\ell}{4}\right)^2 = \frac{7}{48}\,m\,\ell^2 = \frac{7}{48}\,(10)\,(1^2) = 1.4583\ \text{kg-m}^2 \qquad (2)$$



Using given data, Eq. (1) can be expressed as

$$1.4583\,\ddot\theta + \frac{1}{16}\,(500)\,(1^2)\,\dot\theta + \frac{5}{8}\,(1000)\,(1^2)\,\theta = \frac{3}{4}\,(1000)\,(1)\,(0.01)\sin 10\,t$$

$$\text{i.e.,}\quad 1.4583\,\ddot\theta + 31.25\,\dot\theta + 625.0\,\theta = 7.5\sin 10\,t \tag{3}$$

Steady state angular displacement of the bar is given by Eq. (3.28) with:

$$\Theta = \frac{7.5}{\left\{\left[625.0 - 1.4583\,(10^2)\right]^2 + 31.25^2\,(10^2)\right\}^{\frac{1}{2}}} = 0.01311\text{ rad}$$

$$\phi = \tan^{-1}\left(\frac{31.25\,(10)}{625.0 - 1.4583\,(10^2)}\right) = 0.5779\text{ rad}$$

$$\therefore\ \theta(t) = \Theta\sin(\omega\,t - \phi) = 0.01311\sin(10\,t - 0.5779)\text{ rad}$$

3.60 Displacement transmissibility (T):

$$T = \frac{X}{Y} = \left[\frac{1 + (2\,\zeta\,r)^2}{(1 - r^2)^2 + (2\,\zeta\,r)^2}\right]^{\frac{1}{2}}$$

For maximum of T,

$$\frac{dT}{dr} = \frac{1}{2}\left[\frac{1 + 4\,\zeta^2\,r^2}{(1 + r^4 - 2\,r^2) + 4\,\zeta^2\,r^2}\right]^{-\frac{1}{2}}$$

$$\frac{\left[(1 - r^2)^2 + (2\,\zeta\,r)^2\right](8\,\zeta^2\,r) - (1 + 4\,\zeta^2\,r^2)\left[4\,r^3 - 4\,r + 8\,\zeta^2\,r\right]}{\left[(1 - r^2)^2 + (2\,\zeta\,r)^2\right]^2} = 0$$

This equation can be simplified to obtain:

$$(2\,\zeta^2)\,r^4 + r^2 - 1 = 0$$

$$\text{Solution:}\quad r^2 = \frac{-1 \pm \sqrt{1 + 8\,\zeta^2}}{4\,\zeta^2}$$

$$\text{or}\quad r = r_m = \frac{1}{2\,\zeta}\sqrt{\sqrt{1 + 8\,\zeta^2} - 1}$$

3.61

Empty

$$m = \frac{1000}{32.2} = 31.0559 \ \frac{lb-sec^2}{ft}$$

At speed v = 55 mph, and wave length 12 ft,

$$\omega = 2\pi f = 2\pi\left(\frac{v\ (1760*3)}{3600}\right)\frac{1}{12}$$

$$= 42.2371 \text{ rad/sec}$$

$$k = 30000 \text{ lb/ft}$$

$$\zeta = 0.2$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{30000}{31.0559}}$$

$$= 31.0805 \text{ rad/sec}$$

$$r = \frac{\omega}{\omega_n} = \frac{42.2371}{31.0805}$$

$$= 1.3590$$

$$(2\zeta r)^2 = (2*0.2*1.3590)^2$$

$$= 0.2955$$

$$(1-r^2)^2 = (1-1.3590^2)^2$$

$$= 0.7170$$

$$\frac{X}{Y} = \left\{\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}\right\}^{\frac{1}{2}}$$

$$= \left\{\frac{1+0.2955}{0.7170+0.2955}\right\}^{\frac{1}{2}}$$

$$= 1.1311$$

Amplitude of vibration of automobile is magnified by a factor of 1.1311

Fully Loaded

$$m = \frac{3000}{32.2} = 93.1677 \ \frac{lb-sec^2}{ft}$$

$$\omega = 42.2371 \text{ rad/sec}$$

$$k = 30000 \text{ lb/ft}$$

$$\zeta = 0.2$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{30000}{93.1677}}$$

$$= 17.9444 \text{ rad/sec}$$

$$r = \frac{\omega}{\omega_n} = \frac{42.2371}{17.9444}$$

$$= 2.3538$$

$$(2\zeta r)^2 = (2*0.2*2.3538)^2$$

$$= 0.8864$$

$$(1-r^2)^2 = (1-2.3538^2)^2$$

$$= 20.6141$$

$$\frac{X}{Y} = \left\{\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}\right\}^{\frac{1}{2}}$$

$$= \left\{\frac{1+0.8864}{20.6141+0.8864}\right\}^{\frac{1}{2}}$$

$$= 0.2962$$

Amplitude of vibration of automobile is diminished by a factor of 0.2962

3.63

Equation of motion: $M\ddot{x} + c\dot{x} + kx = m e \omega^2 \sin \omega t$

where $\omega = \dfrac{3000\,(2\pi)}{60} = 314.16$ rad/sec, $M = 100$ kg, $c = 2000$ N-s/m, $k = 10^6$ N/m, $m = 0.1$ kg and $e = r = 0.1$ m. Steady state response is:

$$x_p(t) = X \sin(\omega t - \phi)$$

$$\text{where}\quad X = \frac{m e \omega^2}{\left[(k - M\omega^2)^2 + (c\,\omega)^2\right]^{\frac{1}{2}}}$$

$$= \frac{0.1\,(0.1)\,(314.16^2)}{\left[\left\{10^6 - 100\,(314.16^2)\right\}^2 + (2000\,(314.16))^2\right]^{\frac{1}{2}}} = 110.9960\,(10^{-6})\text{ m}$$

$$\text{and}\quad \phi = \tan^{-1}\left(\frac{c\,\omega}{k - M\omega^2}\right) = \tan^{-1}\left(\frac{2000\,(314.16)}{10^6 - 100\,(314.16^2)}\right)$$

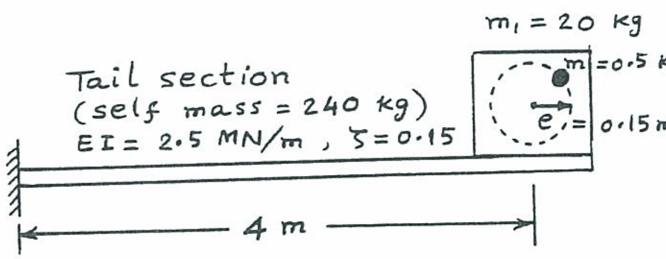
$$= -0.07072\text{ rad} = -4.0520^\circ$$

3.64

k = spring constant of cantilever beam

$$= \frac{3EI}{l^3} = \frac{3\,(2.5\times10^6)}{4^3}$$

$$= 0.1172\times10^6\text{ N/m}$$



$$\omega_n = \sqrt{\frac{k}{m_1 + 0.25\,m_b}} = \sqrt{\frac{0.1172\times10^6}{20 + 0.25\,(240)}} = 38.2753\text{ rad/sec}$$

$$\omega = 2\pi\,(1500)/60 = 157.08\text{ rad/sec}$$

$$r = \omega/\omega_n = 157.08/38.2753 = 4.1040\,, \qquad r^2 = 16.8428$$

Forced response is given by Eq. (3.79):

$$x_p(t) = X \sin(\omega t - \phi)$$

where

$$X = \frac{m e}{m_1}\cdot\frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

$$= \frac{(0.5)(0.15)}{20}\;\frac{16.8428}{\sqrt{(1 - 16.8428)^2 + (2\times0.15\times4.1040)^2}}$$

$$= 3.9747\times10^{-3}\text{ m} = 3.9747\text{ mm}$$

3.65 $\delta_{st} = \frac{45}{1000}\ m = \frac{Mg}{k} = \frac{380 \times 9.81}{k}$

i.e., $k = 82,840$ N/m

$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{82,840}{380}} = 14.7648$ rad/sec ; $\omega = \frac{2\pi(1750)}{60} = 183.26$ rad/sec

$r = \frac{\omega}{\omega_n} = \frac{183.26}{14.7648} = 12.412$; $r^2 = 154.0566$

(i) Amplitude of vibration

$$X = \frac{me}{M} \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{0.15}{380} \frac{154.0566}{\sqrt{(153.0566)^2 + 0}}$$

$= 3.9732 \times 10^{-4}$ m

(ii) Force transmitted to ground

$= kX = (82840)(3.9732 \times 10^{-4}) = 32.9140$ N

3.66 $I = \frac{1}{12}(0.5)(0.1)^3 = 0.4167 \times 10^{-4}\ m^4$

$$k = \frac{192\,EI}{l^3} = \frac{192\,(2.07 \times 10^{11})(0.4167 \times 10^{-4})}{(5)^3} = 1.3248 \times 10^7 \text{ N/m}$$

(a) $\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{13.248 \times 10^6}{75}} = 420.2856$ rad/sec

$\omega = 2\pi(1200)/60 = 125.664$ rad/sec

$r = \omega/\omega_n = 125.664/420.2856 = 0.299$, $r^2 = 0.0894$

Amplitude of steady-state vibration is given by Eq. (3.30) with $\zeta = 0$:

$$X = \frac{\delta_{st}}{|r^2 - 1|} = \frac{F_0}{k|(r^2-1)|} = \frac{5000}{(1.3248 \times 10^7)(0.9106)}$$

$= 0.4145 \times 10^{-3}$ m

(b) Using the effective mass due to self weight of beam (for a cantilever) to be valid here also,

$$\omega_n = \sqrt{\frac{k}{M + 0.2357\,m}}$$

where M = mass of motor = 75 kg, and

m = mass of beam = $(5 \times 0.5 \times 0.1)\left(\frac{76.5 \times 10^3}{9.81}\right) = 1949.5313$ kg

$$\omega_n = \sqrt{\frac{13.248 \times 10^6}{75 + (1949.5313)(0.2357)}} = 157.4339 \text{ rad/sec}$$

$r = \omega/\omega_n = 125.664/157.4339 = 0.7982$, $r^2 = 0.6371$

$$X = \frac{\delta_{st}}{|r^2 - 1|} = \frac{F_0}{k\,|r^2-1|} = \frac{5000}{(1.3248 \times 10^7)(0.3629)}$$

$$= 1.0400 \times 10^{-3} \text{ m}$$

3.67 Let width $= 0.5$ m and thickness $= t$ m.

$$I = \frac{1}{12}(0.5)\,t^3 = \frac{t^3}{24} \text{ m}^4$$

$$k = \frac{3EI}{l^3} = \frac{3(2.07 \times 10^{11})(t^3/24)}{(5)^3} = 2.07 \times 10^8\, t^3 \text{ N/m}$$

$$\omega_n = \sqrt{\frac{k}{M + 0.2357\, m}}$$

where m = mass of beam $= (5 \times 0.5 \times t)\left(\frac{76.5 \times 10^3}{9.81}\right) = 19495.41\,t$ kg

$$\omega_n = \sqrt{\frac{2.07 \times 10^8\, t^3}{75 + 0.2357\,(19495.41\, t)}}$$

$$r = \frac{\omega}{\omega_n} = 125.664 \sqrt{\frac{75 + 4595.0688\, t}{2.07 \times 10^8\, t^3}}$$

$$X = \frac{\delta_{st}}{|r^2 - 1|} = \frac{F_0}{k\,|r^2 - 1|}$$

i.e.,

$$0.5 = \frac{5000}{(2.07 \times 10^8\, t^3)\left\{(125.664)^2\left[\frac{75 + 4595.0688\,t}{2.07 \times 10^8\, t^3}\right] - 1\right\}}$$

i.e., $1.3108 \times 10^4\, t^3 - 4595.069\, t - 74.367 = 0$

By trial and error, the value of t is found as

$$t \approx 0.6 \text{ m}.$$

Since this is too large, we can start with a new width such as 1.0 m.

3.68 $m = (600/9.81)$ N , $\omega = 2\pi(1000)/60 = 104.72$ rad/sec

$k = 6\,(6000) = 36{,}000$ N/m

$$\omega_n = \sqrt{k/m} = \sqrt{36000\Big/\left(\frac{600}{9.81}\right)} = 24.2611 \text{ rad/sec}$$

$$r = \omega/\omega_n = 104.72/24.2611 = 4.3164 \,, \quad r^2 = 18.6311$$

$$X = \frac{F_0}{k\,|r^2-1|} = \frac{m_0\, e\, \omega^2}{k\,|r^2 - 1|}$$

where m_0 = unbalanced mass and e = eccentricity

i.e., $$2.5 \times 10^{-3} = \frac{m_o e \, (104.72)^2}{36000 \, |17.6311|}$$

i.e., $m_o e = 0.1447$ kg-m

$\therefore$ Unbalance $= W_o e = m_o g e = 0.1447 (9.81) = 1.4195$ N-m

3.69 $m = \frac{1000}{386.4} = 2.588 \ \frac{lb-s^2}{in}$, $\quad \omega = \frac{2\pi (1500)}{60} = 157.08 \ \frac{rad}{s}$

Possible isolators are:
(i) $k = 45\,000$ lb/in, $\zeta = 0$
(ii) $k = 90\,000$ lb/in, $\zeta = 0$
(iii) $k = 45\,000$ lb/in, $\zeta = 0.15$
(iv) $k = 90\,000$ lb/in, $\zeta = 0.15$

We will compare the force transmissibilities of these isolators.

Force transmissibility $= T_r = \sqrt{\dfrac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}}$

(i) $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{45000}{2.588}} = 131.8634$ rad/sec

$r = \omega/\omega_n = 157.08/131.8634 = 1.1912$, $\quad r^2 = 1.4190$

$T_r = \dfrac{1}{|1-r^2|} = \dfrac{1}{0.419} = 2.3866$

(ii) $\omega_n = \sqrt{\frac{90\,000}{2.588}} = 186.4829$ rad/sec

$r = \omega/\omega_n = 157.08/186.4829 = 0.8423$, $\quad r^2 = 0.7095$

$T_r = \dfrac{1}{|1-r^2|} = \dfrac{1}{0.2905} = 3.4423$

(iii) $\omega_n = \sqrt{\frac{45\,000}{2.588}} = 131.8634$ rad/sec

$r = 1.1912$, $\quad r^2 = 1.4190$, $\quad \zeta = 0.15$

$$T_r = \sqrt{\frac{1 + (2 \times 1.1912 \times 0.15)^2}{(1 - 1.4190)^2 + (2 \times 1.1912 \times 0.15)^2}} = 1.9282$$

(iv) $\omega_n = 186.4829$ rad/sec, $\quad r = 0.8423$, $\quad r^2 = 0.7095$, $\quad \zeta = 0.15$

$$T_r = \sqrt{\frac{1 + (2 \times 0.8423 \times 0.15)^2}{(1 - 0.7095)^2 + (2 \times 0.8423 \times 0.15)^2}} = 2.6789$$

$\therefore$ Isolation (iii) is best.

3.70 Eq. (3.82):

$$\frac{MX}{me} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

When $r = 1$,

$$\frac{MX}{me} = \frac{1}{2\zeta} \quad \text{or} \quad \frac{M}{me} = \frac{1}{2\zeta X} = \frac{1}{2\zeta(0.55)} = \frac{1}{1.1\zeta} \qquad (E_1)$$

When r = large,

$$\frac{MX}{me} \approx 1 \quad \text{or} \quad \frac{M}{me} \approx \frac{1}{X} = \frac{1}{0.15} \qquad (E_2)$$

Combining (E_1) and (E_2), we obtain

$$\frac{M}{me} = \frac{1}{0.15} = \frac{1}{1.1\zeta}$$

$$\therefore \quad \zeta = 0.1364$$

3.71 For each spring,

$$\underset{\sim}{k} = \frac{G d^4}{64\, n\, R^3} = \frac{(11.5385 \times 10^6)(0.25)^4}{64\,(8)\,(1.5)^3} = 26.083 \text{ lb/in}$$

Total $k = 4\,(26.083) = 104.332$ lb/in

$$\omega = \frac{2\pi\,(1800)}{60} = 188.496 \text{ rad/sec}$$

$m = 100/386.4$ lb-s²/in , $M = 750/386.4$ lb-s²/in , $\zeta = 0$

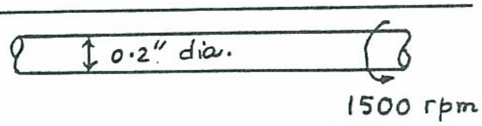
$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{104.332}{(750/386.4)}} = 7.3316 \text{ rad/sec}$$

$r = 188.496/7.3316 = 25.7102$, $\quad r^2 = 661.0144$

$$X = \frac{me}{M} \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{100\,(0.01)}{750} \left(\frac{661.0144}{660.0144}\right)$$

$$= 1.3354 \times 10^{-3} \text{ in.}$$

3.72 $\omega = 2\pi\,(1500)/60 = 157.08$ rad/sec



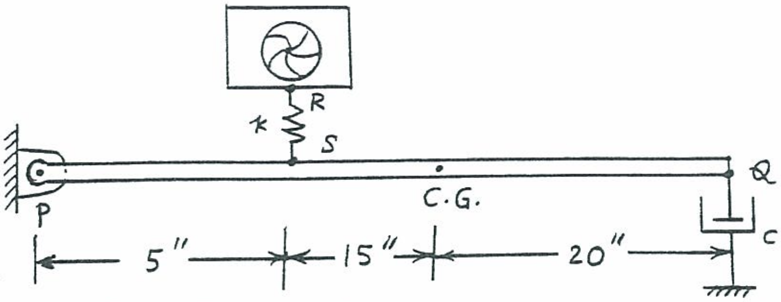
(a) Force due to eccentricity of rotor

$$= m e \omega^2 = \left(\frac{30}{386.4}\right)(0.01)(157.08)^2 = 19.1569 \text{ lb.}$$

(b) H.P. = (Force)(eccentricity)(angular velocity)

$$= (19.1569)\left(\frac{0.01}{12}\right)\left(\frac{157.08}{550}\right) = 0.004559 \text{ hp.}$$

3.74



$$\omega_{n,fan} = \sqrt{\frac{k}{m_{fan}}} = \sqrt{\frac{200}{50/386.4}} = 39.3141 \text{ rad/sec}$$

$$\omega = \frac{2\pi(750)}{60} = 78.54 \text{ rad/sec}$$

$$(J_P)_{plate+fan} = \frac{1}{3}\left(\frac{100}{386.4}\right)(40)^2 + \left(\frac{50}{386.4}\right)(5)^2 = 141.2612 \text{ lb-in-sec}^2$$

$$F_0 = m e \omega^2 = \left(\frac{50}{386.4}\right)(0.1)(78.54)^2 = 79.8205 \text{ lb}$$

Point R is subjected to the force, $F(t) = F_0 \cos \omega t = 79.8205 \cos 78.54\,t$ lb.

Assume that S is not moving.

Then R is displaced by:

$$x(t) = \frac{F_0 \cos \omega t}{|k - m\omega^2|} = \frac{F_0 \cos \omega t}{k\left|1 - \left(\frac{\omega}{\omega_n}\right)^2\right|} = \frac{79.8205 \cos \omega t}{200\left|1 - \left(\frac{78.54}{39.3141}\right)^2\right|}$$

$$= 0.1334 \cos 78.54\,t \text{ inch}$$

Let θ = angular displacement of plate PQ.

Displacement of $S = 5\,\theta$ inch

Extension of spring RS $= (5\theta - 0.1334 \cos 78.54\,t)$ inch

Restoring moment of spring force about P

$$= 200\,[5\,\theta - 0.1334 \cos 78.54\,t]\,5 \text{ lb-in}$$

Velocity of $Q = 40\,\dot{\theta}$ inch/sec

Damping force at $Q = 40\,\dot{\theta}\,(1) = 40\,\dot{\theta}$ lb

Moment of damping force about $P = 40\,\dot{\theta}\,(40) = 1600\,\dot{\theta}$ lb-in

Equation of motion of plate PQ:

$$J_P\,\ddot{\theta} + 1600\,\dot{\theta} + 1000\,(5\theta - 0.1334 \cos 78.54\,t) = 0$$

i.e., $$141.2612\,\ddot{\theta} + 1600\,\dot{\theta} + 5000\,\theta = 133.4 \cos 78.54\,t \qquad (E_1)$$

Comparing (E_1) with Eq. (3.24), the solution of (E_1) can be expressed as $\theta_p(t) = \Theta \cos(\omega t - \phi)$

where, from Eqs. (3.30) and (3.31), we get

$$\Theta = \frac{(133.4/5000)}{\sqrt{(1 - 174.2751)^2 + (2 \times 0.9519 \times 13.2013)^2}} = 1.5238 \times 10^{-4} \text{ rad}$$

and $\phi = \tan^{-1}(-25.1326/173.2751) = -8.2529°$

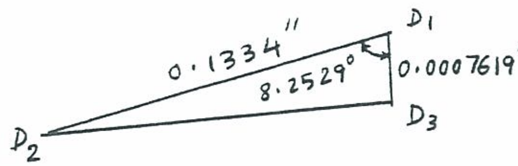
Steady state motion of $Q = \theta_p\,(40)$

$$= 0.006095 \cos(78.54\,t + 8.2529°) \text{ inch}$$

Displacement of $S = \Theta(5)$ inch

$= (1.5238 \times 10^{-4})(5)$ inch

$= 0.0007619$ inch



$D_2 D_3 =$ maximum deformation of spring $\approx 0.1334''$

Max. force transmitted to point $S = k(D_2 D_3)$

$= 200(0.1334) = 26.68$ lb

3.78

$$I = \int_0^{2\pi/\omega} \sin \omega t \cdot \cos(\omega t - \phi)\, dt$$

$$= \int_0^{2\pi/\omega} \sin \omega t \left[\cos \omega t \cdot \cos \phi + \sin \omega t \cdot \sin \phi \right] dt$$

$$= \int_0^{2\pi/\omega} \left\{ \cos \phi \left(\sin \omega t \cdot \cos \omega t \right) + \sin \phi \left(\sin^2 \omega t \right) \right\} dt$$

$$= \int_0^{2\pi/\omega} \left\{ \cos \phi \left(\frac{\sin 2\omega t}{2} \right) + \sin \phi \left(\frac{1 - \cos 2\omega t}{2} \right) \right\} dt$$

$$= \frac{\cos \phi}{2} \left(- \frac{\cos 2\omega t}{2\omega} \right) \Bigg|_0^{2\pi/\omega} + \frac{\sin \phi}{2} \left(t - \frac{\sin 2\omega t}{2} \right) \Bigg|_0^{2\pi/\omega}$$

$$= \frac{\pi}{\omega} \sin \phi$$

$$\Delta W' = \omega F_0 X \cdot I = \omega F_0 X \sin \phi$$

3.79

Let $x(t)$ = displacement of mass m

New length of each spring, $k_1 = \left(l^2 + x^2 \right)^{1/2}$

New tension in each spring $k_1 = T = \left(\sqrt{l^2 + x^2} - l \right) k_1 + T_0$

Horizontal component of new tension in each spring k_1

$$= T x \big/ \sqrt{l^2 + x^2}$$

Vertical component of new tension in each spring $k_1 = \dfrac{T \cdot l}{\sqrt{l^2 + x^2}}$

Total friction force $= \mu\, mg + \dfrac{2 T l}{\sqrt{l^2 + x^2}}$

when mass moves to right:

Equation of motion of mass m:

$$m \ddot{x} + k_2 x + \frac{2 T x}{\sqrt{l^2 + x^2}} - \mu \left[mg + \frac{2 T l}{\sqrt{l^2 + x^2}} \right] = p_0 A \sin \omega t$$

where A = area of piston.

i.e., $\quad m \ddot{x} + x \left(k_2 + 2 \dfrac{T_0}{l} \right) = \mu mg + 2 \mu T_0 + p_0 A \sin \omega t$

Similarly, when the mass moves to left:

$$m \ddot{x} + x \left(k_2 + 2 \frac{T_0}{l} \right) = -\mu mg - 2 \mu T_0 + p_0 A \sin \omega t$$

3.82 $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2100}{2}} = 32.403703$ rad/sec

N = vertical force = $mg = 2\,(9.81) = 19.62$ N

$$\frac{\omega}{\omega_n} = \frac{2.5173268 \times 2\pi}{32.403703} = 0.4881191$$

$$X = \frac{F_0}{k}\left[\frac{1 - \left(\frac{4\mu N}{\pi F_0}\right)^2}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2}\right]^{1/2}$$

i.e., $$0.075 = \frac{120}{2100}\left[\frac{1 - \left\{\frac{4\mu(19.62)}{\pi(120)}\right\}^2}{(1 - 0.4881191^2)}\right]^{1/2}$$

i.e., $$1.3125 = \left(\frac{1 - 0.04334\,\mu^2}{0.5802473}\right)^{1/2}$$

i.e., $$0.9995666 = 1 - 0.04334\,\mu^2$$

i.e., $$\mu = 0.1$$

3.83 (a) $k = \frac{W}{\delta_{st}} = \frac{5000}{0.05} = 10^5$ N/m

When $\omega = \omega_n$, Eq. (3.102) gives

$$X = \frac{F_0}{k\beta} \Rightarrow 0.1 = \frac{1000}{(10^5)\,\beta} \Rightarrow \beta = 0.1$$

(b) $\Delta W = \pi\, c_{eq}\, \omega\, X^2 = \pi \beta k X^2$ where $c_{eq} = \frac{\beta k}{\omega}$ from Eq. (3.100)

$$\Delta W = \pi\,(0.1)\,(10^5)\,(0.1)^2 = 314.16 \text{ Joules/cycle}$$

(c) Steady state amplitude at one-quarter of resonant frequency:

$$\frac{\omega}{\omega_n} = 0.25$$

$$X = \frac{F_0}{k\left[\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \beta^2\right]^{1/2}} = \frac{1000}{10^5\left[\{1 - 0.25^2\}^2 + (0.1)^2\right]^{1/2}}$$

$$= 0.01061 \text{ m}$$

(d) Steady state amplitude at thrice the resonant frequency:

$$\frac{\omega}{\omega_n} = 3$$

$$X = \frac{1000}{10^5\left[(1 - 3^2)^2 + (0.1)^2\right]^{1/2}} = 0.00125 \text{ m}$$

3.84

$$\Delta W = \pi \beta k X^{\gamma}$$

$$\left.\begin{aligned} 3.8 &= \pi \beta (60\,000)(0.04)^{\gamma} \\ 9.5 &= \pi \beta (60000)(0.06)^{\gamma} \end{aligned}\right\} \Rightarrow \quad \begin{aligned} \beta (0.04)^{\gamma} &= 0.0000202 \\ \beta (0.06)^{\gamma} &= 0.0000504 \end{aligned}$$

Taking logarithms,

$$\ln \beta + \gamma \ln (0.04) = \ln (0.0000202)$$

$$\ln \beta + \gamma \ln (0.06) = \ln (0.0000504)$$

i.e.,

$$\ln \beta - 3.218876\, \gamma = -10.809828 \quad ---- (E_1)$$

$$\ln \beta - 2.813411\, \gamma = -9.895511 \quad ---- (E_2)$$

subtracting (E_1) from (E_2), $\quad 0.405465\, \gamma = 0.914309$

$$\gamma = 2.254964$$

From (E_1), $\quad \ln \beta = -10.809828 + 3.218876\,(2.254964) = -3.551378$

$$\beta = 0.028685$$

3.85

Work done $= W = \int F\, dx = \int F\, \dot{x}\, dt$

If $F(t) = F_0 \cos \omega t$ and $x(t) = X \cos(\omega t - \phi)$, work done in one cycle

$$= W = -\int_0^{2\pi/\omega} F_0 \cos \omega t .\ \omega X \sin(\omega t - \phi)\, dt$$

$$= -\frac{F_0 \omega X \cos \phi}{2} \left(-\frac{1}{2\omega} \cos 2\omega t \right)_0^{2\pi/\omega} + \frac{F_0 \omega X \sin \phi}{2} \left(t + \frac{1}{2\omega} \sin 2\omega t \right)_0^{\frac{2\pi}{\omega}}$$

$$= F_0 \pi X \sin \phi$$

Given data: $\quad F_0 = 5\ lb$, $\omega = 3\pi\ \frac{rad}{sec}$, $\tau = \frac{2}{3}$ sec, $\phi = \frac{\pi}{3}$, $X = 0.5''$

$$W = F_0 \pi X \sin \phi = 5\pi (0.5) \sin \frac{\pi}{3} = 6.8018\ lb\text{-}in$$

(i) In one second, it will complete $1\frac{1}{2}$ cycles.

$$W\Big|_{1\ \text{second}} = 1.5\ W = 10.2027\ lb\text{-}in.$$

(ii) In four seconds, it will complete 6 cycles.

$$W\Big|_{4\ \text{seconds}} = 6\ W = 40.8108\ lb\text{-}in.$$

3.86

Damping force $= F = c\,(\dot{x})^n$

Energy dissipated per quarter cycle during harmonic motion $x(t) = X \sin \omega t$ is

$$\frac{\Delta W}{4} = \int_0^{\pi/2\omega} c\,(\dot{x})^n\, dx = \int_0^{\pi/2\omega} c\,(\omega X \cos \omega t)^n\, dx$$

But $\quad dx = \dot{x}\, dt = \omega X \cos \omega t \cdot dt$

$$\Delta W = 4c\,\omega^{n+1}\,X^{n+1} \int_0^{\pi/2\omega} \cos^{n+1}\omega t \cdot dt$$

$$= 4c\,\omega^{n+1}\,X^{n+1} \left\{ \frac{1}{(n+1)\omega} \cos^n \omega t \cdot \sin \omega t \Big|_0^{\pi/2\omega} + \frac{n}{n+1} \int_0^{\pi/2\omega} \cos^{n-1} \omega t \cdot dt \right\}$$

$$= 4c\,\omega^{n+1}\,X^{n+1} \left(\frac{n}{n+1}\right) \int_0^{\pi/2\omega} \cos^{n-1}\omega t \cdot dt$$

Equating this expression to $\pi\, c_{eq}\, \omega\, X^2$, we obtain

$$c_{eq} = \frac{4c\,\omega^n\,X^{n-1}}{\pi} \left(\frac{n}{n+1}\right) \int_0^{\pi/2\omega} \cos^{n-1}\omega t \cdot dt \equiv c\,\omega^n\,X^{n-1}\,\alpha_n$$

where $$\alpha_n = \frac{4}{\pi}\left(\frac{n}{n+1}\right) \int_0^{\pi/2\omega} \cos^{n-1}\omega t \cdot dt \quad ---- (E_1)$$

For example, for $n = 2$, (E_1) becomes

$$\alpha_n = \frac{4}{\pi}\left(\frac{2}{3}\right) \int_0^{\pi/2\omega} \cos\omega t \cdot dt = \frac{8}{3\pi}\left(\frac{\sin \omega t}{\omega}\right)_0^{\pi/2\omega} = \frac{8}{3\pi\omega}$$

and hence $$c_{eq} = \frac{8\,c\,\omega\,X}{3\pi}$$

which can be seen to be same as the expression found in Example 3.7.

For few other values of n, α_n can be found as follows:

n	1	2	3	4
α_n	$\frac{1}{\omega}$	$\frac{8}{3\pi\omega}$	$\frac{3}{4\omega}$	$\frac{32}{15\pi\omega}$

The amplitude can be found as

$$X = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + c_{eq}^2\,\omega^2}} = \frac{F_0}{\sqrt{k^2(1-r^2)^2 + c_{eq}^2\,\omega^2}}$$

$$= \frac{F_0}{\sqrt{k^2(1-r^2)^2 + c^2\,\omega^{2(n+1)}\,X^{2(n-1)}\,\alpha_n^2}}$$

3.87 Energy dissipated per cycle for viscous damping $= \pi c\,\omega X^2$

Energy dissipated per cycle for Coulomb damping $= 4\mu N X$

Equivalent viscous damping constant (c_{eq}) is given by

$$\pi\, c_{eq}\,\omega X^2 = \pi c\,\omega X^2 + 4\mu N X$$

$$c_{eq} = \left(c + \frac{4\mu N}{\pi\omega X}\right)$$

Amplitude X is given by

$$X = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + c_{eq}^2 \omega^2}} = \frac{F_0}{\sqrt{k^2(1-r^2)^2 + c_{eq}^2 \omega^2}}$$

substituting for c_{eq}, squaring and rearranging,

$$X^2 \left\{ k^2(1-r^2)^2 + c^2\omega^2 \right\} + X \left(\frac{8 \mu N c \omega}{\pi} \right) + \left(\frac{16 \mu^2 N^2}{\pi^2} - F_0^2 \right) = 0$$

3.88 (a) Equation of motion $\quad m\ddot{x} \pm \mu N + c(\dot{x})^3 + kx = F_0 \cos \omega t$

Thus the system has combined Coulomb and velocity-cubed damping.

For Coulomb damping, $\quad c_{eq1} = \dfrac{4\mu N}{\pi \omega X} \qquad (E_1)$

For velocity-cubed damping, the equivalent viscous damping coefficient can be obtained from the solution of problem 3.68:

$$c_{eq2} = c\, \omega^3 X^2 \alpha_3 \qquad (E_2)$$

where

$$\alpha_3 = \frac{4}{\pi}\left(\frac{3}{4}\right) \int_0^{\pi/2\omega} \cos^2 \omega t \; dt = \frac{3}{4\omega} \qquad (E_3)$$

$$\therefore \quad c_{eq2} = \frac{3}{4} c\, \omega^2 X^2 \qquad (E_4)$$

and

$$c_{eq} = c_{eq1} + c_{eq2} = \frac{4\mu N}{\pi \omega X} + \frac{3}{4} c\omega^2 X^2 \qquad (E_5)$$

(b) Steady state amplitude under harmonic force:

$$X = \frac{F_0}{\sqrt{k^2(1-r^2)^2 + c_{eq}^2 \omega^2}} = \frac{F_0}{\sqrt{k^2(1-r^2)^2 + \left\{ \frac{4\mu N}{\pi \omega X} + \frac{3}{4} c\omega^2 X^2 \right\}^2 \omega^2}} \qquad (E_6)$$

(c) Amplitude ratio:

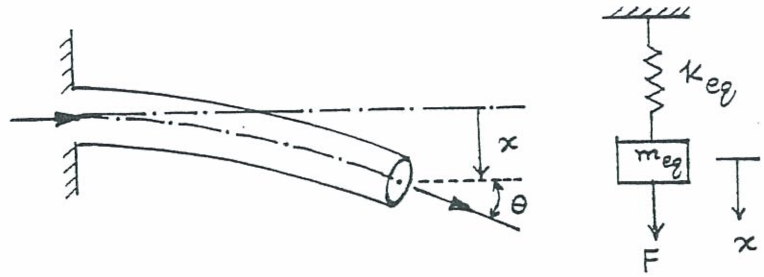
$$\frac{X}{\delta_{st}} = \frac{X}{(F_0/k)} = \frac{1}{\sqrt{(1-r^2)^2 + \left(\frac{c_{eq}^2 \omega^2}{k^2}\right)}}$$

$$= \frac{1}{\sqrt{(1-r^2)^2 + \left\{ \frac{4\mu N}{\pi X k} + \frac{3}{4} \frac{c\, \omega^3 X^2}{k} \right\}^2}} \qquad (E_7)$$

At resonance, $r = 1$ and Eq. (E_7) reduces to

$$\left.\frac{X}{\delta_{st}}\right|_{resonance} = \frac{1}{\left\{ \frac{4\mu N}{\pi X k} + \frac{3}{4k} c\, \omega^3 X^2 \right\}} \qquad (E_8)$$

3.89 Model the pipe as a single degree of freedom system with m_{eq} = equivalent mass at end $= \dfrac{33}{140}$ m (m = mass of pipe; see Problem 2.46) and $k_{eq} = \dfrac{3\,E\,I}{\ell^3}$. Slope of pipe at end:



$$\theta = \frac{F\,\ell^2}{2\,E\,I} = \frac{F\,\ell^3}{3\,E\,I}\left(\frac{3}{2\,\ell}\right) = \frac{3\,x}{2\,\ell}$$

where x = end deflection of the cantilever pipe under a transverse load F. Force induced due to fluid velocity v is $\rho\,A\,v^2$. Force acting on the single degree of freedom system (in vertical direction):

$$F = \rho\,A\,v^2\,\sin\theta \approx \rho\,A\,v^2\,\theta = \rho\,A\,v^2\,\frac{3\,x}{2\,\ell}$$

Equation of motion: $m_{eq}\,\ddot{x} + k_{eq}\,x = F$

or $$\frac{33}{140}\,m\,\ddot{x} + \left(\frac{3\,E\,I}{\ell^3} - \frac{3\,\rho\,A\,v^2}{2\,\ell}\right)x = 0$$

Instability occurs when $\dfrac{3\,E\,I}{\ell^3} - \dfrac{3\,\rho\,A\,v^2}{2\,\ell} < 0$ or $v > \sqrt{\dfrac{2\,E\,I}{\rho\,A\,\ell^2}}$

3.90 Assume Reynolds number (R_e) greater than 1000. Strouhal number (St) for vortex shedding is taken as: $St = \dfrac{f\,d}{V} = 0.21$ where f = frequency of vortex shedding, d = diameter of cylinder and V = velocity of fluid (air). At 50 mph speed,

$$V = \frac{50\,(1760)\,(36)}{3600} = 880 \text{ in/sec} \quad \text{and} \quad f = \frac{0.21\,V}{d} = \frac{184.8}{d}\ \text{Hz} \quad (\text{d in inches})$$

For the three sections of the antenna, the vortex frequencies are:

$$f_1 = \frac{184.8}{0.3} = 616.0\ \text{Hz}$$
$$f_2 = \frac{184.8}{0.2} = 924.0\ \text{Hz}$$
$$f_3 = \frac{184.8}{0.1} = 1848.0\ \text{Hz}$$

At 75 mph speed,

$$V = \frac{75\,(1760)\,(36)}{3600} = 1320 \text{ in/sec} \quad \text{and} \quad f = \frac{0.21\,V}{d} = \frac{277.2}{d}\ \text{Hz} \quad (\text{d in inches})$$

For the three sections of the antenna, the frequencies are:

$$f_1 = \frac{277.2}{0.3} = 924.0\ \text{Hz}$$
$$f_2 = \frac{277.2}{0.2} = 1386.0\ \text{Hz}$$

$$f_3 = \frac{277.2}{0.1} = 2772.0 \text{ Hz}$$

Since the natural frequencies are much smaller, no instability occurs.

3.91 (a) Equivalent mass of single d.o.f. system $= m_{eq} = M + \frac{33}{140}$ m where m = mass of cylindrical part of the sign post:

$$m = \frac{\pi}{4}(D^2 - d^2)\, h\, \rho = \frac{\pi}{4}(0.25^2 - 0.2^2)(10)\left(\frac{76500}{9.81}\right) = 1378.0527 \text{ kg}$$

$$\therefore m_{eq} = 200 + \frac{33}{140}(1378.0527) = 524.8267 \text{ kg}$$

$$I = \frac{\pi}{64}(D^4 - d^4) = \frac{\pi}{4}(0.25^4 - 0.2^4) = 113.208\,(10^{-6})\text{ m}^4$$

Equivalent stiffness of the system:

$$k_{eq} = \frac{3\,E\,I}{h^3} = \frac{3\,(207\,(10^9))\,(113.208\,(10^{-6}))}{10^3} = 70{,}302.168 \text{ N/m}$$

Natural frequency of transverse vibration of sign post:

$$\omega_1 = \left(\frac{k_{eq}}{m_{eq}}\right)^{\frac{1}{2}} = \left(\frac{70302.168}{524.8267}\right)^{\frac{1}{2}} = 11.5738 \text{ rad/sec} = 1.8420 \text{ Hz}$$

(b) Wind velocity corresponding to maximum vibration of sign post (V) is given by:

$$St = 0.21 = \frac{f_1\,D}{V} \quad \text{or} \quad V = \frac{f_1\,D}{0.21} = \frac{(1.8420)\,(0.25)}{0.21} = 2.1929 \text{ m/s}$$

(c) Maximum force acting on the system due to wind velocity:

$$F(t) = F_0 \sin\omega t = \frac{1}{2}\, c\, \rho\, V^2\, A \sin\omega t = \frac{1}{2}(1)(1.2215)(2.19299^2)(8.0)\sin\omega t \text{ N}$$
$$= 23.4958 \sin\omega t \text{ N}$$

where c = 1 for a cylinder, ρ = density of air = 1.2215 kg/m^3, A = projected area of cylindrical part = (0.8)(10) = 8.0 m^2, and ω = frequency of wind force. Equation of motion:

$$m_{eq}\,\ddot{x} + c_{eq}\,\dot{x} + k_{eq}\,x = F(t)$$

and the maximum steady state displacement of the sign post occurs when $\omega = \omega_1$ and is given by Eq. (3.34):

$$X = \frac{\delta_{st}}{2\,\zeta} = \frac{F_0}{k_{eq}\,(2)\,\zeta} = \frac{23.4958}{2\,(0.1)\,(70302.168)} = 0.001671 \text{ m}$$

3.92 (a) Equation of motion $\quad m\ddot{x} + c\dot{x} + kx = F_0 x$

or $\quad m\ddot{x} + c\dot{x} + (k - F_0)\, x = 0 \qquad (E_1)$

Assuming the solution $\quad x(t) = C\, e^{st} \qquad (E_2)$

where C is a constant, Eq. (E_1) gives the auxiliary equation

$$s^2 + \frac{c}{m} s + \left(\frac{k - F_0}{m}\right) = 0 \qquad (E_3)$$

Roots of (E_3) are

$$s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k - F_0}{m}\right)} \qquad (E_4)$$

First consider the case of positive stiffness $(k > F_0)$. For this case, following possibilities exist.

1. If $\left(\frac{c}{2m}\right)^2 > \left(\frac{k - F_0}{m}\right)$:

   Both s_1 and s_2 will be real and negative and hence

   $$x(t) = C_1\, e^{s_1 t} + C_2\, e^{s_2 t} \qquad (E_5)$$

   will be stable.

2. If $\left(\frac{c}{2m}\right)^2 = \left(\frac{k - F_0}{m}\right)$:

   Both s_1 and s_2 will be identical, real and negative.

   Solution $\quad x(t) = (C_1 + C_2 t)\, e^{s_1 t} \qquad (E_6)$

   will be stable since $e^{s_1 t} \to 0$ as $t \to \infty$.

3. If $\left(\frac{c}{2m}\right)^2 < \left(\frac{k - F_0}{m}\right)$:

   Here s_1 and s_2 will be complex conjugates and solution will be

   $$x(t) = C\, e^{-\left(\frac{c}{2m}\right)t} \sin\left(\sqrt{\left\{-\left(\frac{c}{2m}\right)^2 + \left(\frac{k - F_0}{m}\right)\right\}}\; t + \phi\right) \qquad (E_7)$$

   This represents a converging oscillatory motion and hence the system will be stable.

Next consider the case of negative stiffness $(k < F_0)$. Here

$$s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 + \left(\frac{F_0 - k}{m}\right)} \qquad (E_8)$$

Thus s_1 will be positive and s_2 will be negative, and the solution becomes

$$x(t) = C_1\, e^{+s_1 t} + C_2\, e^{-|s_2| t} \qquad (E_9)$$

This solution can be seen to diverge as $t \to \infty$.

(b) Equation of motion $\quad m\ddot{x} + c\dot{x} + kx = F_0 \dot{x}$

or $\quad \ddot{x} + \left(\frac{c - F_0}{m}\right)\dot{x} + \frac{k}{m} x = 0 \qquad (E_{10})$

Assuming $x(t) = C\, e^{st}$ the auxiliary equation becomes

$$s^2 + \left(\frac{c - F_0}{m}\right) s + \frac{k}{m} = 0 \qquad (E_{11})$$

and hence

$$s_{1,2} = -\left(\frac{c - F_0}{2m}\right) \pm \sqrt{\left(\frac{c - F_0}{2m}\right)^2 - \frac{k}{m}} \qquad (E_{12})$$

First consider the case of positive damping $(c > F_0)$ in (E_{10}). For this case, it can be seen that the system will be stable for all possible values of $\left\{ \left(\frac{c - F_0}{2m}\right)^2 - \frac{k}{m} \right\}$.

Next, consider the case of negative damping $(c < F_0)$. Depending on the sign of the quantity under the radical in Eq. (E_{12}), we will have three types of solution.

1. $\left(\frac{c - F_0}{2m}\right)^2 > \frac{k}{m}$. Here both s_1 and s_2 are real and positive and hence $x(t) = C_1\, e^{s_1 t} + C_2\, e^{s_2 t}$ (E_{13})

   This denotes a diverging nonoscillatory motion; so the system is unstable.

2. $\left(\frac{c - F_0}{2m}\right)^2 = \frac{k}{m}$. Here s_1 and s_2 are identical and are real and positive. Hence $x(t) = (C_1 + C_2\, t)\, e^{s_1 t}$ (E_{14})

   This represents a diverging nonoscillatory solution; so the system will be unstable.

3. $\left(\frac{c - F_0}{2m}\right)^2 < \frac{k}{m}$. Here s_1 and s_2 are complex conjugates and hence

$$s_{1,2} = \left(\frac{F_0 - c}{2m}\right) \pm i \sqrt{\frac{k}{m} - \left(\frac{c - F_0}{2m}\right)^2} \qquad (E_{15})$$

   The solution becomes

$$x(t) = X\, e^{\left(\frac{F_0 - c}{2m}\right) t} \sin\left(\sqrt{\frac{k}{m} - \left(\frac{c - F_0}{2m}\right)^2}\; t + \phi \right) \qquad (E_{16})$$

   Since the exponent is positive, Eq. (E_{16}) denotes a diverging oscillatory motion and hence the system is unstable.

Thus the condition for dynamic stability of the system can be stated as

$$F_0 \le c \qquad (E_{17})$$

(c) Equation of motion $\quad m\ddot{x} + c\dot{x} + k x = F_0 \ddot{x}$

or $\quad (m - F_0)\ddot{x} + c\dot{x} + k x = 0 \qquad (E_{18})$

with the solution $\quad x(t) = C\, e^{st} \qquad (E_{19})$

the auxiliary equation will be

$$s^2 + \left(\frac{c}{m - F_0}\right) s + \left(\frac{k}{m - F_0}\right) = 0 \qquad (E_{20})$$

The roots are

$$s_{1,2} = -\frac{c}{2(m - F_0)} \pm \sqrt{\left\{\frac{c}{2(m - F_0)}\right\}^2 - \left(\frac{k}{m - F_0}\right)} \qquad (E_{21})$$

First consider the case of positive mass $(m > F_0)$ in (E_{18}). In this case, the system will be stable for all values of

$$\left[\left\{\frac{c}{2(m - F_0)}\right\}^2 - \left(\frac{k}{m - F_0}\right)\right].$$

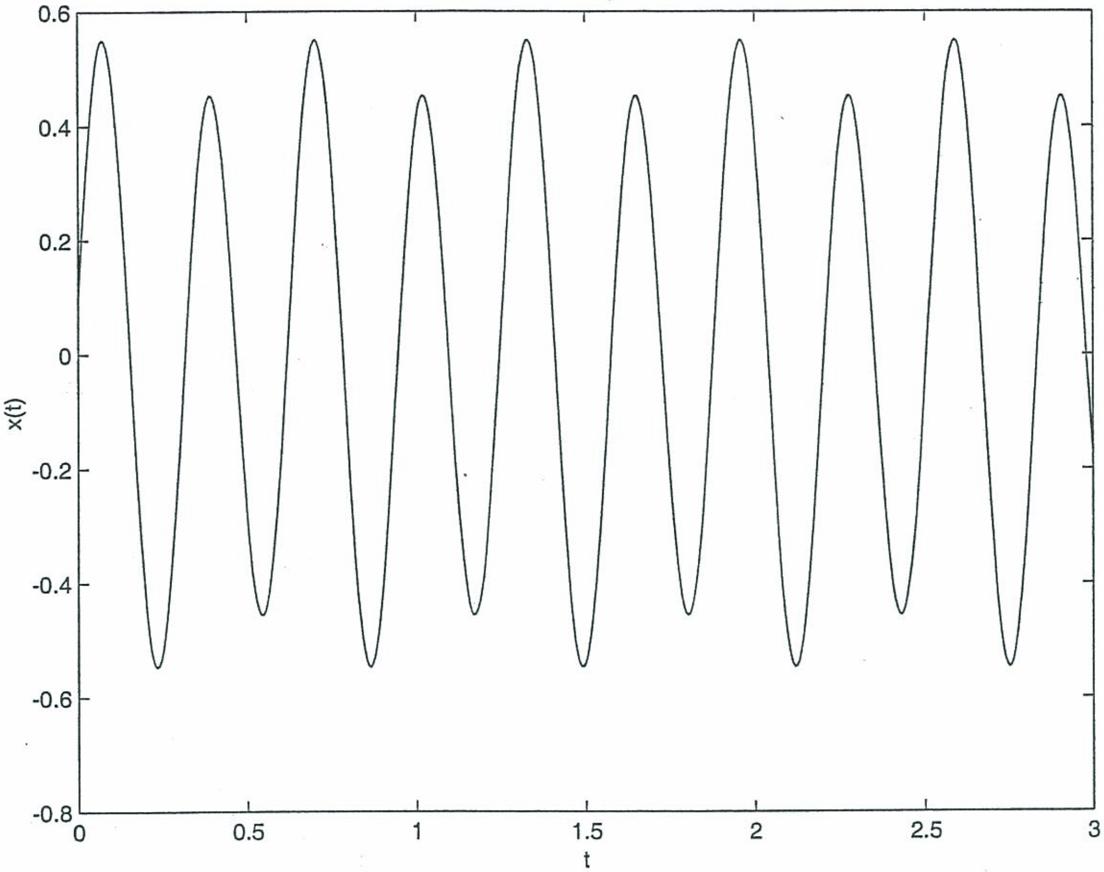
Next consider the case of negative mass $(m < F_0)$ in (E_{18}). For this case s_1 and s_2 can be expressed as

$$s_{1,2} = \frac{c}{2(F_0 - c)} \pm \sqrt{\left\{\frac{c}{2(F_0 - m)}\right\}^2 + \left(\frac{k}{F_0 - m}\right)} \qquad (E_{22})$$

This shows that s_1 will be positive and s_2 will be negative ; thus the solution will be divergent.

3.103

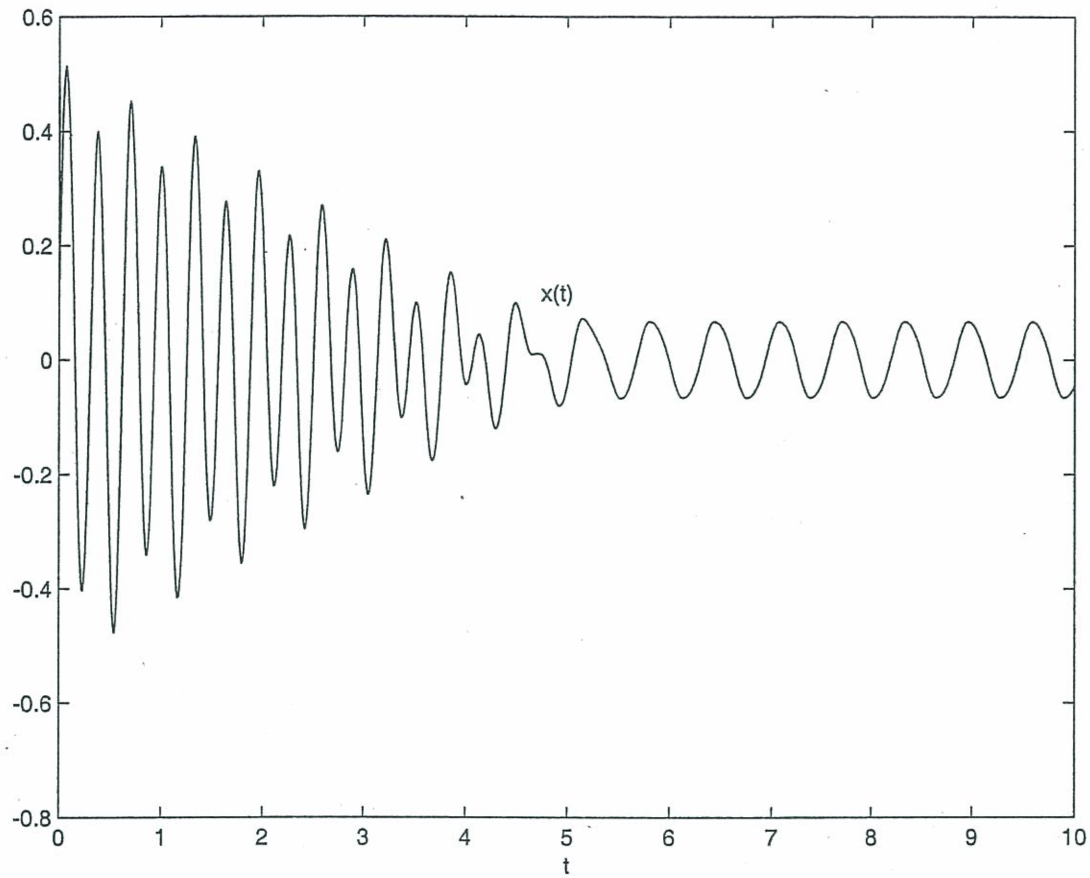
```
% Ex3_103.m
k = 4000;
m = 10;
w = 10;
F0 = 200;
wn = sqrt(k/m);
x0 = 0.1;
x0_dot = 10;
f_0 = F0/m;
for i = 1: 501
    t(i) = 3 * (i-1)/500;
    x(i) = x0_dot*sin(wn*t(i))/wn + (x0 - f_0/(wn^2-w^2))*cos(wn*t(i))...
        + f_0/(wn^2-w^2)*cos(w*t(i));
end
plot(t,x);
xlabel('t');
ylabel('x(t)');
```



3.104

```
% Ex3_104.m
% This program will use the function dfunc3_104.m,they should
% be in the same folder
tspan = [0: 0.01: 10];
x0 = [0.1; 10];
[t,x] = ode23('dfunc3_104', tspan, x0);
disp('       t          x(t)      xd(t)');
disp([t x]);
plot(t,x(:,1));
xlabel('t');
gtext('x(t)');

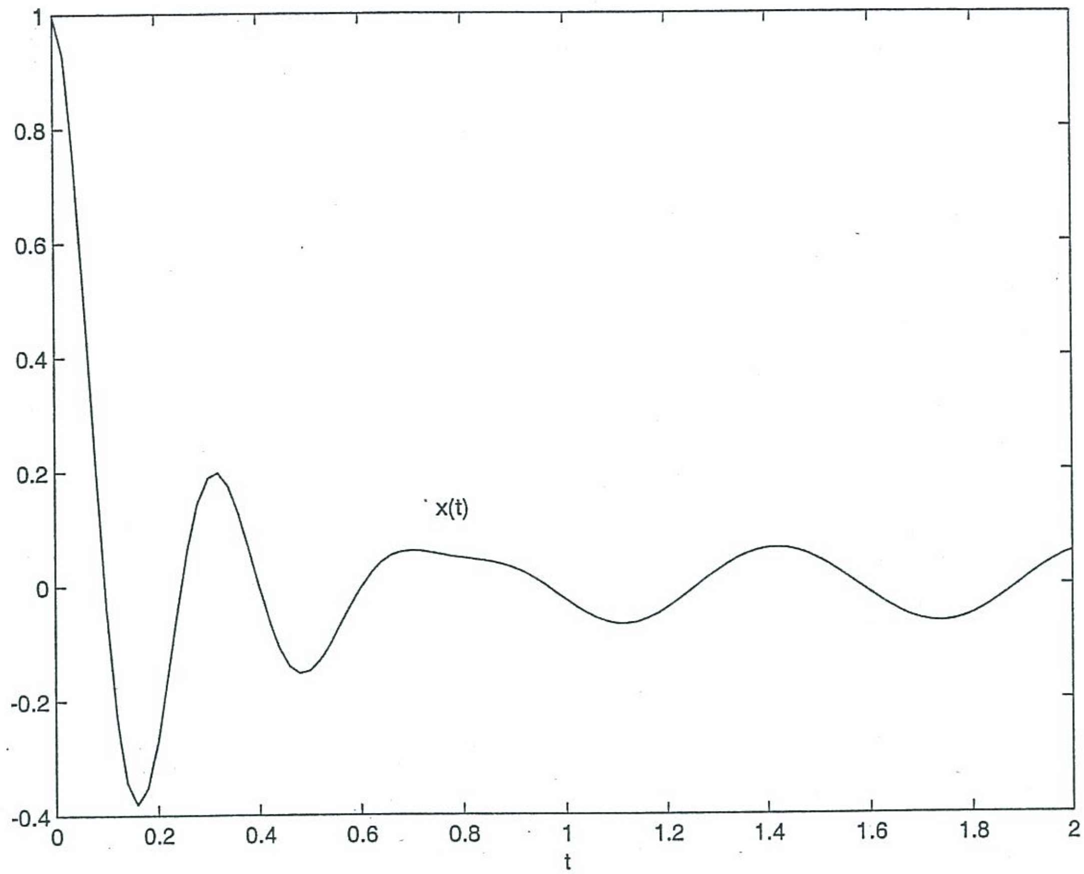
% dfunc3_104.m
function f = dfunc3_104(t,x)
F0 = 200;
w = 10;
u = 0.3;
m = 10;
k = 4000;
f = zeros(2,1);
f(1) = x(2);
f(2) = (F0/m)*sin(w*t) - 9.81*u*sign(x(2)) - (k/m)*x(1);
```



3.105

```
% Ex3_105.m
% This program will use the function dfunc3_105.m, they should
% be in the same folder
tspan = [0: 0.02: 2];
x0 = [1; 0];
[t,x] = ode23('dfunc3_105', tspan, x0);
disp('      t         x(t)      xd(t)');
disp([t x]);
plot(t,x(:,1));
xlabel('t');
gtext('x(t)');
```

```
% dfunc3_105.m
function f = dfunc3_105(t,x)
m = 100;
k = 40000;
zeta = 0.25;
Y = 0.05;
w = 10;
c = 2 * zeta * sqrt(k*m);
f = zeros(2,1);
f(1) = x(2);
f(2) = k*Y*sin(w*t)/m + c*w*Y*cos(w*t)/m - c*x(2)/m - (k/m)*x(1);
```




```
>> Ex3_105
```

t	x(t)	xd(t)
0	1.0000	0
0.0200	0.9272	-6.9328
0.0400	0.7381	-11.5448
0.0600	0.4823	-13.6094
0.0800	0.2096	-13.3072
0.1000	-0.0372	-11.1218
0.1200	-0.2270	-7.7195
•		
•		
•		
1.8000	-0.0523	0.3904
1.8200	-0.0434	0.4869
1.8400	-0.0329	0.5637
1.8600	-0.0210	0.6179
1.8800	-0.0083	0.6473
1.9000	0.0047	0.6510
1.9200	0.0176	0.6286
1.9400	0.0297	0.5811
1.9600	0.0406	0.5103
1.9800	0.0499	0.4194
2.0000	0.0572	0.3120

3.106

```
%===================================================================
%
% Ex3_106.m(Program3.m)
% Main program which calls HARESP
%
%===================================================================
% Run "Ex3_106.m" in MATLAB command window. Ex3_106.m and haresp.m should
% be in the same folder,and set the path to this folder
% following seven lines contain problem-dependent data
xm=10.0;
xk=1000;
zeta=0.1;
xc=2*zeta*sqrt(xk*xm);
f0=100.0;
om=20.0;
n=20;
ic=1;
% end of problem-dependent data
[t,x,xd,xdd,xamp,xphi]=haresp(xm,xc,xk,f0,om,ic,n);
% following lines output the results
fprintf('Steady state response of an undamped\n');
fprintf('Single degree of freedom system under harmonic force\n\n');
fprintf('Given data\n');
fprintf('xm = %10.8e\n',xm);
fprintf('xc = %10.8e\n',xc);
fprintf('xk = %10.8e\n',xk);
fprintf('f0 = %10.8e\n',f0);
fprintf('om = %10.8e\n',om);
fprintf('ic = %1.0f\n',ic);
fprintf('n  = %2.0f\n\n\n',n);
fprintf('Response: \n\n');
fprintf('   i          x(i)                    xd(i)              xdd(i)');
fprintf('\n\n');
for i=1:n
   fprintf('  %2.0f    %10.8e          %10.8e     %10.8e\n',i,x(i),...
       xd(i),xdd(i));
end
subplot(311);
plot(t,x);
ylabel('x(t)');
gtext('x(t)');
subplot(312);
plot(t,xd);
ylabel('xd(t)');
gtext('xd(t)');
subplot(313);
plot(t,xdd);
ylabel('xdd(t)');
gtext('xdd(t)');
xlabel('t');
```

```
%=====================================================================
%
%function haresp.m
%
%=====================================================================
function [t,x,xd,xdd,xamp,xphi]=haresp(xm,xc,xk,f0,om,ic,n);
omn=sqrt(xk/xm);
xai=xc/(2.0*xm*omn);
dst=f0/xk;
r=om/omn;
xamp=dst/sqrt((1.0-r^2)^2+(2.0*xai*r)^2);
xphi=atan(2.0*xai*r/(1.0-r^2));
delt=2.0*3.1416/(om*n);
time=0.0;
if ic~=0
   for i=1:n
      time=time+delt;
      t(i) = time;
      x(i)=xamp*cos(om*time-xphi);
      xd(i)=-xamp*om*sin(om*time-xphi);
      xdd(i)=-xamp*om^2*cos(om*time-xphi);
   end
else
   for i=1:n
      time=time+delt;
      t(i) = time;
      x(i)=xamp*sin(om*time-xphi);
      xd(i)=xamp*om*cos(om*time-xphi);
      xdd(i)=-xamp*om^2*sin(om*time-xphi);
   end
end
```

```
>> Ex3_106
Steady state response of an undamped
Single degree of freedom system under harmonic force

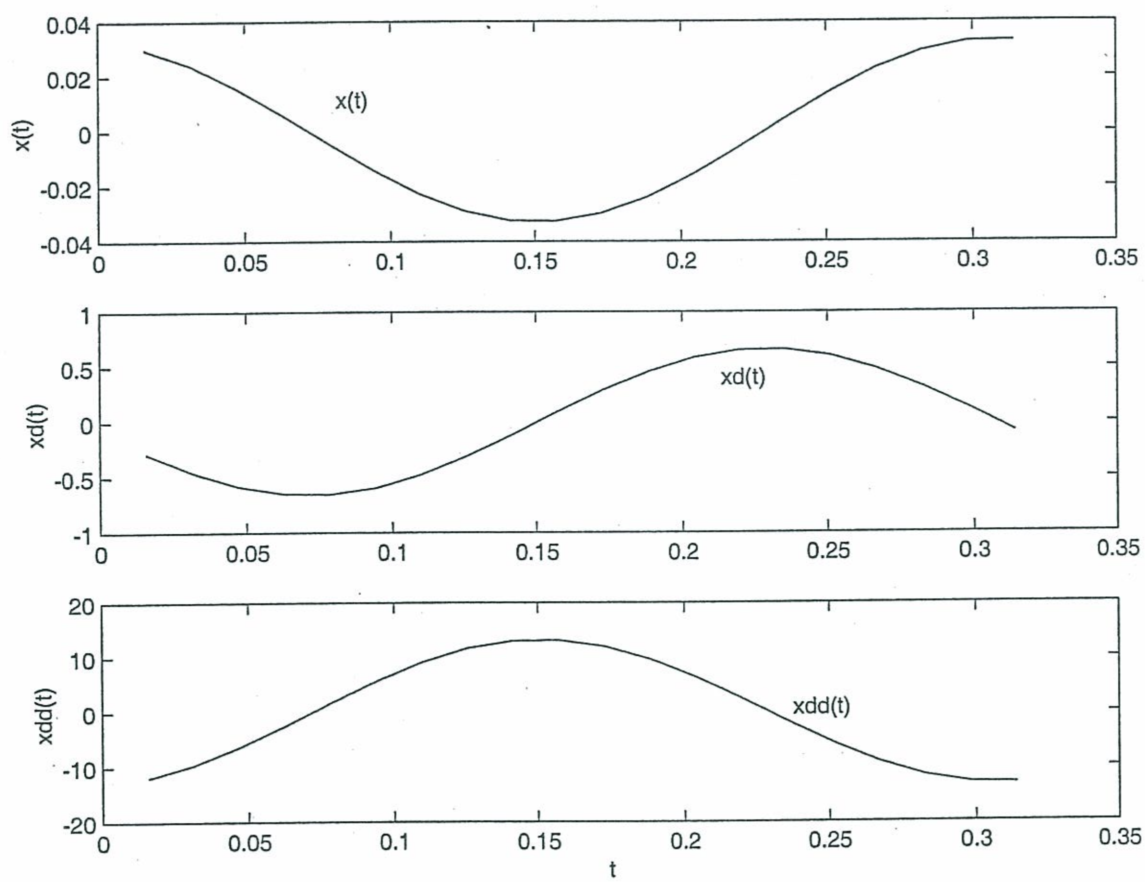
Given data
xm = 1.00000000e+001
xc = 2.00000000e+001
xk = 1.00000000e+003
f0 = 1.00000000e+002
om = 2.00000000e+001
ic = 1
n  = 20

Response:

   i        x(i)                 xd(i)               xdd(i)

   1     2.97987095e-002      -2.85475021e-001     -1.19194838e+001
   2     2.39294085e-002      -4.55669383e-001     -9.57176339e+000
   3     1.57177193e-002      -5.81259445e-001     -6.28708774e+000
   4     5.96746320e-003      -6.49951518e-001     -2.38698528e+000
   5    -4.36693253e-003      -6.55021513e-001      1.74677301e+000
   6    -1.42738605e-002      -5.95973141e-001      5.70954420e+000
```

7	-2.27835571e-002	-4.78586496e-001	9.11342282e+000
8	-2.90630300e-002	-3.14352252e-001	1.16252120e+001
9	-3.24975978e-002	-1.19346877e-001	1.29990391e+001
10	-3.27510596e-002	8.73410566e-002	1.31004238e+001
11	-2.97986046e-002	2.85479399e-001	1.19194419e+001
12	-2.39292411e-002	4.55672899e-001	9.57169644e+000
13	-1.57175058e-002	5.81261754e-001	6.28700233e+000
14	-5.96722446e-003	6.49952395e-001	2.38688978e+000
15	4.36717313e-003	6.55020872e-001	-1.74686925e+000
16	1.42740794e-002	5.95971044e-001	-5.70963176e+000
17	2.27837329e-002	4.78583148e-001	-9.11349314e+000
18	2.90631454e-002	3.14347982e-001	-1.16252582e+001
19	3.24976417e-002	1.19342102e-001	-1.29990567e+001
20	3.27510275e-002	-8.73458687e-002	-1.31004110e+001



3.107

```
%Ex3_107.m
Y= 0.05;
zeta = 0.1;
wn = 8.164966;
w = 2.90889;
r = w/wn;
x = Y * sqrt( (1 + (2 * zeta * r)^2)/((1 - r^2)^2 + (2 * zeta * r)^2) );
disp('   w                    wn                    x');
disp([w wn x]);
w = 14.54445;
r = w/wn;
x = Y * sqrt( (1 + (2 * zeta * r)^2)/((1 - r^2)^2 + (2 * zeta * r)^2) );
disp([w wn x]);
w = 29.08890;
r = w/wn;
x = Y * sqrt( (1 + (2 * zeta * r)^2)/((1 - r^2)^2 + (2 * zeta * r)^2) );
disp([w wn x]);
wn = 6.324555;
w = 2.90889;
r = w/wn;
x = Y * sqrt( (1 + (2 * zeta * r)^2)/((1 - r^2)^2 + (2 * zeta * r)^2) );
disp([w wn x]);
w = 14.54445;
r = w/wn;
x = Y * sqrt( (1 + (2 * zeta * r)^2)/((1 - r^2)^2 + (2 * zeta * r)^2) );
disp([w wn x]);
w = 29.08890;
r = w/wn;
x = Y * sqrt( (1 + (2 * zeta * r)^2)/((1 - r^2)^2 + (2 * zeta * r)^2) );
disp([w wn x]);
```

```
>> Ex3_107
   w                    wn                    x
   2.90889000000000    8.16496600000000    0.05722376420338

  14.54445000000000    8.16496600000000    0.02410324256879

  29.08890000000000    8.16496600000000    0.00524102723160

   2.90889000000000    6.32455500000000    0.06325355032007

  14.54445000000000    6.32455500000000    0.01275990975243

  29.08890000000000    6.32455500000000    0.00336736169683
```

3.108 Equation of motion is $m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky$

when $y(t) = Y \sin \omega t$, $x_p(t) = X \cos(\omega t - \phi_1 - \phi_2)$

Complete solution can be expressed as

$$x(t) = X_0\, e^{-\zeta\omega_n t} \cos(\omega_d t + \phi_0) + X \cos(\omega t - \phi_1 - \phi_2) \qquad (E.1)$$

with $X = Y \left[\dfrac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2} \right]^{1/2}$,

$$\phi_1 = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right), \quad \phi_2 = \tan^{-1}\left(\frac{1}{2\zeta r}\right), \quad r = \frac{\omega}{\omega_n}.$$

If the initial conditions are known $x(t=0) = x_0$ and $\dot{x}(t=0) = \dot{x}_0$,

$$x_0 = X_0 \cos\phi_0 + X \cos(\phi_1 + \phi_2)$$

and

$$\dot{x}_0 = -\zeta\omega_n X_0 \cos\phi_0 - \omega_d X_0 \sin\phi_0 - \omega X \sin(-\phi_1 - \phi_2)$$

Hence

$$X_0 = \left[\{x_0 - X\cos(\phi_1+\phi_2)\}^2 + \left\{ \frac{-\dot{x}_0 - \zeta\omega_n x_0 + \zeta\omega_n X \cos(\phi_1+\phi_2) + \omega X \sin(\phi_1+\phi_2)}{\omega_d} \right\}^2 \right]^{1/2}$$

$$\phi_0 = \tan^{-1}\left[\frac{-\dot{x}_0 - \zeta\omega_n x_0 + \zeta\omega_n X\cos(\phi_1+\phi_2) + \omega X \sin(\phi_1+\phi_2)}{\omega_d \{x_0 - X\cos(\phi_1+\phi_2)\}} \right]$$

If necessary, the velocity $\dot{x}(t)$ and acceleration $\ddot{x}(t)$ can be found from Eq. (E.1). The computer program and output are given below.

```
C ==================================================================
C
C SOLUTION OF PROBLEM 3.108
C MAIN PROGRAM WHICH CALLS BASEX
C RESPONSE OF A SINGLE D.O.F. SYSTEM SUBJECTED TO BASE EXCITATION,
C Y(T)=Y*SIN(OM*T)
C
C ==================================================================
```

```
C FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION X(20),XD(20),XDD(20)
      DATA XM,XC,XK,Y,OM,N/2.0,10.0,100.0,0.1,25.0,20/
      DATA X0,XD0/0.01,5.0/
C END OF PROBLEM-DEPENDENT DATA
      CALL BASEX (XM,XC,XK,Y,OM,N,X,XD,XDD,X0,XD0)
      PRINT 100
 100  FORMAT (//,33H TOTAL RESPONSE OF AN UNDERDAMPED,/,
     2 52H SINGLE D.O.F. SYSTEM UNDER HARMONIC BASE EXCITATION)
      PRINT 200, XM,XC,XK,Y,OM,N
 200  FORMAT (//,12H GIVEN DATA:,/,5H XM =,E15.8,/,5H XC =,E15.8,/,
     2 5H XK =,E15.8,/,5H Y  =,E15.8,/,5H OM =,E15.8,/,5H N  =,I2)
      PRINT 300,X0,XD0
 300  FORMAT (/,20H INITIAL CONDITIONS:,/,6H X0  =,E15.8,/,6H XD0 =,
     2 E15.8)
      PRINT 400
 400  FORMAT (//,10H RESPONSE:,//,5H   I  ,3X,5H X(I),12X,6H XD(I),
     2 11X,7H XDD(I),/)
      DO 500 I=1,N
 500  PRINT 600, I,X(I),XD(I),XDD(I)
 600  FORMAT (I4,2X,E15.8,2X,E15.8,2X,E15.8)
      STOP
      END
C ====================================================================
C
C SUBROUTINE BASEX
C
C ====================================================================
      SUBROUTINE BASEX (XM,XC,XK,Y,OM,N,X,XD,XDD,X0,XD0)
      DIMENSION X(N),XD(N),XDD(N)
      OMN=SQRT(XK/XM)
      XAI=XC/(2.0*XM*OMN)
      OMD=OMN*SQRT(1.0-XAI**2)
      R=OM/OMN
      DELT=2.0*3.1416/(OMD*REAL(N))
      XAMP=Y*SQRT(1.0+(2.0*XAI*R)**2/((1.0-R**2)**2+(2.0*XAI*R)**2))
      PHI1=ATAN(2.0*XAI*R/(1.0-R*R))
      PHI2=ATAN(1.0/(2.0*XAI*R))
      XCC=XC
      TIME=0.0
      DO 10 I=1,N
      TIME=TIME+DELT
      XC=X0-XAMP*COS(PHI1+PHI2)
      XS=(-XD0-XAI*OMN*XC+OM*XAMP*SIN(PHI1+PHI2))/OMD
      XZ=SQRT(XC**2+XS**2)
      PZ=ATAN(XS/XC)
      EX=EXP(-XAI*OMN*TIME)
      CS=COS(OMD*TIME+PZ)
      SI=SIN(OMD*TIME+PZ)
      CS12=COS(OM*TIME-PHI1-PHI2)
      SI12=SIN(OM*TIME-PHI1-PHI2)
      X(I)=XZ*EX*CS+XAMP*CS12
      XD(I)=-XAI*OMN*XZ*EX*CS-OMD*XZ*EX*SI-OM*XAMP*SI12
 10   XDD(I)=XZ*EX*CS*((XAI*OMN)**2-OMD**2)+2.0*XAI*OMN*OMD*XZ*EX*SI
     2 -(OM**2)*XAMP*CS12
```

```
      XC=XCC
      RETURN
      END
```

```
TOTAL RESPONSE OF AN UNDERDAMPED
SINGLE D.O.F. SYSTEM UNDER HARMONIC BASE EXCITATION


GIVEN DATA:
XM = 0.20000000E+01
XC = 0.10000000E+02
XK = 0.10000000E+03
Y  = 0.10000000E+00
OM = 0.25000000E+02
N  =20

 INITIAL CONDITIONS:
 XO  = 0.99999998E-02
 XDO = 0.50000000E+01


 RESPONSE:
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I	X(I)	XD(I)	XDD(I)
1	-0.50330855E-01	-0.57580528E+01	-0.10305269E+02
2	-0.30772516E+00	-0.44899216E+01	0.62558521E+02
3	-0.43412885E+00	-0.65008521E+00	0.85064461E+02
4	-0.38465756E+00	0.22720275E+01	0.28117821E+02
5	-0.27407524E+00	0.18295681E+01	-0.40405914E+02
6	-0.24066399E+00	-0.41657627E+00	-0.39765869E+02
7	-0.28432682E+00	-0.90879965E+00	0.23411983E+02
8	-0.27970907E+00	0.14345939E+01	0.64174423E+02
9	-0.14624044E+00	0.38781688E+01	0.26183165E+02
10	0.39102390E-01	0.33305802E+01	-0.47345394E+02
11	0.12840362E+00	0.26650119E+00	-0.67513680E+02
12	0.80883794E-01	-0.18088942E+01	-0.10970811E+02
13	0.96553117E-02	-0.68976790E+00	0.50778744E+02
14	0.38462169E-01	0.18209940E+01	0.40712753E+02
15	0.14735566E+00	0.22107751E+01	-0.27525837E+02
16	0.19996722E+00	-0.31552225E+00	-0.66972618E+02
17	0.11764607E+00	-0.28227291E+01	-0.26468327E+02
18	-0.18114097E-01	-0.23140993E+01	0.45068512E+02
19	-0.63696094E-01	0.51316518E+00	0.59701672E+02
20	0.10478826E-01	0.21295214E+01	0.43720150E+00

3.111 Unbalanced force in vertical direction $= m e \omega^2 \sin \omega t$ $\quad (E_1)$

Unbalanced force in horizontal direction $= 0$

Let M = total mass of the shaker

Equation of motion is $\quad M\ddot{x} + c\dot{x} + kx = m e \omega^2 \sin \omega t \quad (E_2)$

Steady state solution of (E_2) is

$$x(t) = X \sin(\omega t - \phi) \qquad (E_3)$$

where

$$X = \frac{m\ e\ r^2}{M\left[(1-r^2)^2 + (2\zeta r)^2\right]^{1/2}} \qquad (E_4)$$

and

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) \qquad (E_5)$$

where

$$r = \frac{\omega}{\omega_n} = \omega\sqrt{(M/k)} \qquad (E_6)$$

Frequency range: 20 Hz to 30 Hz

i.e., 125.664 rad/sec to 188.496 rad/sec

$$\therefore \quad 125.664 \text{ rad/sec} \le \omega \le 188.496 \text{ rad/sec} \qquad (E_7)$$

$0.1'' \le X \le 0.2''$ where X is given by (E_4).

Mean power output over a time period τ is given by

$$P = \frac{1}{\tau}\int_0^{\tau} F(\tau)\ \frac{dx}{dt}(\tau)\ d\tau \qquad (E_9)$$

where $\tau = \frac{2\pi}{\omega}$,

$F(\tau) = m e \omega^2 \sin \omega t$, and

$\frac{dx}{dt} = \omega X \cos(\omega t - \phi)$

$$P \ge 1 \text{ hp} \qquad (E_{10})$$

$$\frac{M}{m} \ge 50 \qquad (E_{11})$$

Procedure:

Find ω, e, M, m, k and c
so as to satisfy the requirements stated in
(E_7), (E_8), (E_{10}) and (E_{11}).

3.112 $m = \frac{10^5}{386.4} = 258.7992$ lb-s²/in , $l = 600''$, $E = 30\times 10^6$ psi

$$k = \frac{3EI}{l^3} = \frac{3\,(30\times 10^6)}{(600)^3}\cdot\frac{\pi}{64}(D^4 - d^4) = 0.020453\,(D^4 - d^4)\ \frac{lb}{in}$$

$\omega = 2\pi(15) = 94.248$ rad/sec ; $\zeta = 0.15$

Ground acceleration $\ddot{y}(t) = 193.2 \sin 94.248\, t$ in/s². $\qquad (E_1)$

Equation of motion:

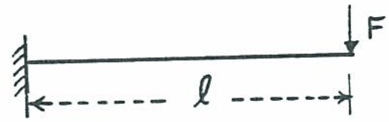
$$m\ddot{z} + c\dot{z} + k z = - m\ddot{y} = -50\,000 \sin 94.248\, t \qquad (E_2)$$

where $z = x - y$ = relative displacement.

Let $$z(t) = Z \sin(\omega t - \phi) = Z \sin(94.248\, t - \phi) \qquad (E_3)$$

with $$Z = \frac{-50\,000}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} \qquad (E_4)$$

and $$\phi = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right) \qquad (E_5)$$



$$y_{max} = \frac{F l^3}{3EI} \Rightarrow \frac{3\, y_{max}}{l^2} = \frac{F l}{EI}$$

Max. bending moment = $M = F l$

Max. bending stress = $\dfrac{M\,(D/2)}{EI} = \dfrac{F l}{EI} \cdot \dfrac{D}{2} = \dfrac{3D}{2 l^2}\, y_{max}$

If maximum relative displacement, $y_{max} = Z$, is known, max. bending stress (σ_1) is given by

$$\sigma_1 = \frac{3D}{2 l^2} \cdot Z$$

Direct compressive stress (σ_2) due to weight of water tank is

$$\sigma_2 = \frac{mg}{\frac{\pi}{4}(D^2 - d^2)} = \frac{4 \times 10^5}{\pi (D^2 - d^2)}$$

Total stress = $\sigma_1 + \sigma_2 \le 30\,000$ psi

i.e., $$\frac{3D}{2 l^2} Z + \frac{4 \times 10^5}{\pi (D^2 - d^2)} \le 30\,000 \qquad (E_6)$$

Natural frequency of water tank is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{0.020453\,(D^4 - d^4)}{258.7992}} \ge 30\pi \ \frac{\text{rad}}{\text{sec}}$$

i.e., $$D^4 - d^4 \ge 1.1240 \times 10^8 \qquad (E_7)$$

Weight of steel column is

$$W_s = \frac{\pi}{4}(D^2 - d^2)\, l\, \rho_{weight} = \frac{\pi}{4}(D^2 - d^2)(600)(0.283)$$

$$= 133.3609\, (D^2 - d^2) \text{ lb.} \qquad (E_8)$$

Problem: Find $\begin{Bmatrix} D \\ d \end{Bmatrix}$ to minimize $\overline{W}_s$ subject to restrictions of (E_6) and (E_7). Also use the conditions : $D \geq d$, $D \geq 0$ and $d \geq 0$.

Procedure: Plot the inequalities (E_6) and (E_7) in the $D-d$ space and draw contours of W_s and identify the minimum weight design.

Chapter 4

Vibration Under General Forcing Conditions

4.1 $F(t) = \frac{F_0}{\pi} + \frac{F_0}{2}\sin\omega t - \frac{2F_0}{\pi}\sum_{n=2,4,6,\ldots}\frac{1}{(n^2-1)}\cos n\omega t$ where $\omega = \frac{2\pi}{\tau}$

$$x(t) = \frac{F_0}{\pi k} + \frac{F_0}{2k}\cdot\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\sin(\omega t - \phi_1)$$

$$-\frac{2F_0}{\pi k}\sum_{n=2,4,6,\ldots}\frac{1}{(n^2-1)}\cdot\frac{1}{\sqrt{(1-n^2r^2)^2+(2\zeta n r)^2}}\cos(n\omega t - \phi_n)$$

where $r = \omega/\omega_n$, $\phi_1 = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$ and $\phi_n = \tan^{-1}\left(\frac{2\zeta n r}{1-n^2r^2}\right)$

4.2 From the solution of problem 1.109,

$$F(t) = \begin{cases} (2F_0 t/\tau) \; ; & 0 \le t \le \tau/2 \\ -(2F_0 t/\tau) + 2F_0 \; ; & \tau/2 \le t \le \tau \end{cases}$$

Fourier series representation of $F(t)$ is

$$F(t) = \frac{F_0}{2} - \frac{4F_0}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2}\cos n\omega t$$

$$\therefore \quad x(t) = \frac{F_0}{2k} - \frac{4F_0}{\pi^2 k}\sum_{n=1,3,\ldots}^{\infty}\frac{1}{n^2}\frac{\cos(n\omega t - \phi_n)}{\sqrt{(1-r^2n^2)^2+(2\zeta n r)^2}}$$

where $r = \frac{\omega}{\omega_n}$ and $\phi_n = \tan^{-1}\left(\frac{2\zeta n r}{1-n^2r^2}\right)$.

4.3 $F(t) = \frac{8F_0}{\pi^2}\sum_{n=1,3,5,\ldots}(-1)^{\frac{n-1}{2}}\sin\frac{n\omega t}{n^2}$ where $\omega = \frac{2\pi}{\tau}$

$$x(t) = \frac{8F_0}{\pi^2 k}\sum_{n=1,3,5,\ldots}\frac{1}{n^2}(-1)^{\frac{n-1}{2}}\frac{1}{\sqrt{(1-n^2r^2)^2+(2\zeta n r)^2}}\sin(n\omega t - \phi_n)$$

where $r = \omega/\omega_n$ and $\phi_n = \tan^{-1}\left(\frac{2\zeta n r}{1-n^2r^2}\right)$

4.4 $F(t) = \frac{F_0}{2} + \frac{F_0}{\pi}\sum_{n=1,2,\ldots}^{\infty}\frac{1}{n}\sin n\omega t$ where $\omega = \frac{2\pi}{\tau}$

$$x(t) = \frac{F_o}{2k} + \frac{F_o}{\pi k} \sum_{n=1,2,\ldots}^{\infty} \frac{1}{n} \cdot \frac{1}{\sqrt{(1-n^2r^2)^2 + (2\zeta n r)^2}} \sin(n\omega t - \phi_n)$$

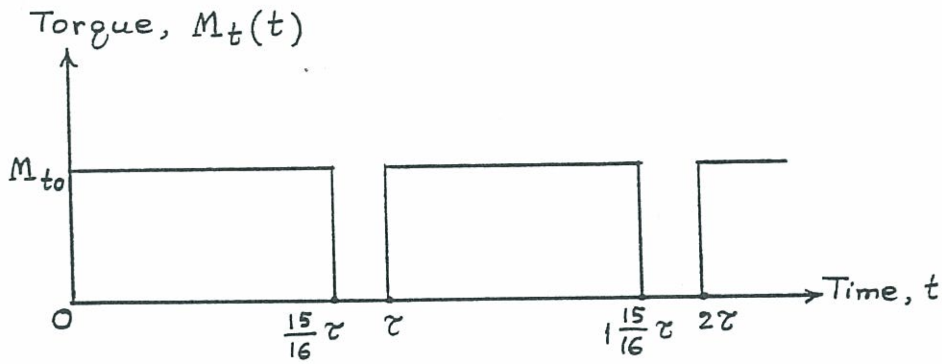
where $r = \omega/\omega_n$ and $\phi_n = \tan^{-1}\left(\frac{2\zeta n r}{1-n^2 r^2}\right)$

4.5 From Example 1.19, $F(t) = \frac{F_o}{\pi}\left[\frac{\pi}{2} - \sum_{n=1,2,3,\ldots} \frac{1}{n} \cdot \sin n\omega t\right]$ where $\omega = \frac{2\pi}{\tau}$

$$x(t) = \frac{F_o}{2k} - \frac{F_o}{\pi k} \sum_{n=1,2,3,\ldots} \frac{1}{n} \cdot \frac{1}{\sqrt{(1-n^2r^2)^2 + (2\zeta n r)^2}} \cdot \sin(n\omega t - \phi_n)$$

where $r = \omega/\omega_n$ and $\phi_n = \tan^{-1}\left(\frac{2\zeta n r}{1-n^2 r^2}\right)$

4.6



Torque transmitted to driven gear is shown in the figure. It can be expressed as:

$$M_t(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos n\,\omega\,t + b_n \sin n\,\omega\,t\right)$$

$$\text{where}\quad \omega = 2\,\pi\left(\frac{1000}{60}\right) = 104.72 \text{ rad/sec}$$

$$\tau = \frac{2\,\pi}{\omega} = 0.06 \text{ sec} \quad ; \quad \frac{15}{16}\,\tau = 0.05625 \text{ sec}$$

$$M_t(t) = \begin{cases} M_{t0} = 1000 \text{ N—m} \;;\; 0 \le t \le \frac{15}{16}\,\tau \\ 0 \;;\; \frac{15}{16}\,\tau \le t \le \tau \end{cases}$$

$$a_0 = \frac{2}{\tau}\int_0^{\tau} M_t(t)\,dt = \frac{2}{0.06}\int_0^{0.05625} (1000)\,dt = \frac{2000}{0.06}(0.05625) = 1875.0 \text{ N—m}$$

$$a_n = \frac{2}{\tau}\int_0^{\tau} M_t(t)\cos n\,\omega\,t\,dt = \frac{2}{\tau} M_{t0}\int_0^{0.05625} \cos n\,\omega\,t\,dt$$

$$= \frac{2\,M_{t0}}{\tau}\left(\frac{\sin n\,\omega\,t}{n\,\omega}\right)_0^{0.05625} = \frac{318.3091}{n} \sin 5.8905\,n \text{ N—m}$$

$$b_n = \frac{2}{\tau}\int_0^\tau M_t(t)\sin n\,\omega\,t\,dt = \frac{2\,M_{t0}}{\tau}\int_0^{0.05625}\sin n\,\omega\,t\,dt$$

$$= \frac{2\,M_{t0}}{\tau}\left(-\frac{\cos n\,\omega\,t}{n\,\omega}\right)_0^{0.05625} = \frac{318.3091}{n}\left(1-\cos 5.8905\,n\right)\ \text{N--m}$$

$$k_t = \frac{GJ}{\ell} = G\left(\frac{\pi\,d^4}{4}\right)\frac{1}{\ell} = (80\,(10^9))\left(\frac{\pi\,(0.05)^4}{4}\right)\frac{1}{1} = 392{,}700\ \text{N--m/rad}$$

$$\omega_n = \sqrt{\frac{k_t}{J_0}} = \sqrt{\frac{392700}{0.1}} = 1981.6660\ \text{rad/sec}$$

Equation of motion:

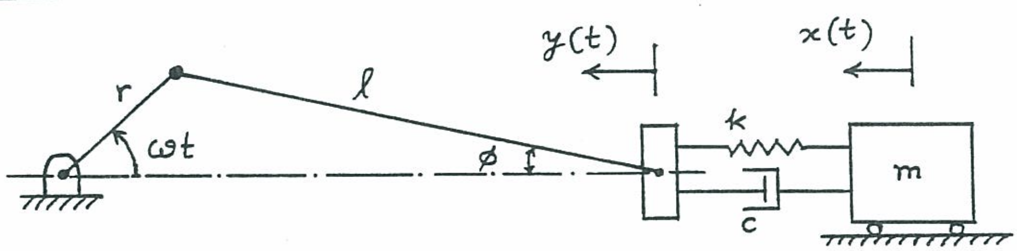
$$J_0\,\ddot\theta + k_t\,\theta = M_t(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos n\,\omega\,t + b_n\sin n\,\omega\,t\right)$$

Response:

$$\theta(t) = \frac{a_0}{2\,k_t} + \sum_{n=1}^{\infty}\left\{\frac{a_n\cos n\,\omega\,t + b_n\sin n\,\omega\,t}{k_t - J_0\,(n\,\omega)^2}\right\}$$

$$= 0.0023873 + \sum_{n=1}^{\infty}\left\{\frac{318.3091\sin 5.8905\,n\cos\omega\,t + 318.3091\,(1-\cos 5.8905\,n)\sin n\,\omega\,t}{n\,(392700.0 - 1096.6278\,n^2)}\right\}\ \text{rad}$$

4.7



Base motion is given by:

$$y(t) = r + \ell - r\cos\omega\,t - \ell\cos\phi = r + \ell - r\cos\omega\,t - \ell\sqrt{1-\sin^2\phi} \tag{1}$$

Using $\ell\sin\phi = r\sin\omega\,t$, Eq. (1) becomes

$$y(t) = r + \ell - r\cos\omega\,t - \ell\sqrt{1 - \frac{r^2}{\ell^2}\sin^2\omega\,t} \tag{2}$$

Using the approximation:

$$\sqrt{1 - \frac{r^2}{\ell^2}\sin^2\omega\,t} \approx 1 - \frac{r^2}{2\,\ell^2}\sin^2\omega\,t \tag{3}$$

Eq. (2) can be expressed as

$$y(t) = r + \ell - r\cos\omega\,t - \ell\left(1 - \frac{1}{2}\frac{r^2}{\ell^2}\sin^2\omega\,t\right)$$

$$= r - r\cos\omega\,t + \frac{\ell}{4}\left(\frac{r}{\ell}\right)^2 - \frac{\ell}{4}\left(\frac{r}{\ell}\right)^2\cos 2\,\omega\,t \tag{4}$$

Equation of motion:

$$m\ddot{x} + c\dot{x} + kx = ky + c\dot{y}$$

$$= kr - kr\cos\omega t + \frac{k\ell}{4}\left(\frac{r}{\ell}\right)^2 - \frac{k\ell}{4}\left(\frac{r}{\ell}\right)^2 \cos 2\omega t + \cdots$$

$$+ c r \omega \sin\omega t + \frac{c\ell}{4}\left(\frac{r}{\ell}\right)^2 (2\omega)\sin 2\omega t + \cdots \tag{5}$$

Solution of Eq. (5) can be found by adding the solutions due to each term on the right hand side of Eq. (5).

Solution due to constant term, F_0 (terms 1 and 3 on the r.h.s. of Eq. (5)):

$$x(t) = \frac{F_0}{k}\left[1 - \frac{1}{\sqrt{1-\varsigma^2}} e^{-\varsigma\omega_n t}\cos(\omega_d t - \phi)\right] \tag{6}$$

$$\text{where} \quad \phi = \tan^{-1}\left(\frac{\varsigma}{\sqrt{1-\varsigma^2}}\right)$$

Solution due to sinusoidal term, $F_0 \sin\Omega t$ (terms 5 and 6 on the r.h.s. of Eq. (5)):

$$x(t) = X\sin(\Omega t - \phi_0) \tag{7}$$

$$\text{where} \quad X = \frac{F_0}{\left[(k - m\Omega^2)^2 + c^2\Omega^2\right]^{\frac{1}{2}}} \quad \text{and} \quad \phi_0 = \tan^{-1}\left(\frac{c\Omega}{k - m\Omega^2}\right) \tag{8}$$

Solution due to cosine term, $F_0 \cos\Omega t$ (terms 2 and 4 in Eq. (5)):

$$x(t) = X\cos(\Omega t - \phi_0) \tag{9}$$

$$\text{where} \quad X = \frac{F_0}{\left[(k - m\Omega^2)^2 + c^2\Omega^2\right]^{\frac{1}{2}}} \quad \text{and} \quad \phi_0 = \tan^{-1}\left(\frac{c\Omega}{k - m\Omega^2}\right) \tag{10}$$

For given data, $\varsigma = \frac{c}{2\sqrt{mk}} = \frac{10}{2\sqrt{1(100)}} = 0.5$, $\frac{r}{\ell} = 0.1$, $\omega = 100$, $2\omega = 200$, etc. and the solution of Eq. (5) can be obtained by using Eqs. (6) to (8) suitably.

4.8 Base motion can be represented by Fourier series as (from Example 1.19):

$$y(t) = \frac{Y}{\pi}\left[\frac{\pi}{2} - \left\{\sin\omega t + \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t + \cdots\right\}\right] \tag{1}$$

Equation of motion of mass:

$$m\,\ddot{x} + c\,(\dot{x} - \dot{y}) + k\,(x - y) = 0 \tag{2}$$

Since y(t) is composed of several terms, the solution of Eq. (2) can be found by superposingg the solutions corresponding to each of the terms appearing in Eq. (1). When $y(t) = Y/2$, constant, equation of motion becomes:

$$m\,\ddot{x} + c\,\dot{x} + k\,x = \frac{k\,Y}{2} = \text{constant} \tag{3}$$

The steady state solution of Eq. (3) is given by (see Example 4.9):

$$x(t) = \frac{Y}{2}\left[1 - \frac{1}{\sqrt{1-\varsigma^2}}\, e^{-\varsigma\,\omega_n\,t} \cos\,(\omega_d\,t - \phi)\right] \tag{4}$$

$$\text{where} \quad \phi = \tan^{-1}\left(\frac{\varsigma}{\sqrt{1-\varsigma^2}}\right) \tag{5}$$

When $y(t) = A \sin \Omega\, t$, the steady state solution of Eq. (2) is given by Eq. (3.67):

$$x(t) = A \sin\,(\Omega\,t - \phi) \tag{6}$$

$$\text{where} \quad A = \left[\frac{1 + (2\,\varsigma\,r)^2}{(1-r^2)^2 + (2\,\varsigma\,r)^2}\right]^{\frac{1}{2}} \tag{7}$$

$$\phi = \tan^{-1}\left(\frac{2\,\varsigma\,r^3}{1 + r^2\,(4\,\varsigma^2 - 1)}\right) \tag{8}$$

$$\text{and} \quad r = \frac{\Omega}{\omega_n}$$

4.9 From solution of Problem 4.7, we can express the base motion as:

$$y(t) = r - r\cos\omega\,t + \frac{\ell}{4}\left(\frac{r}{\ell}\right)^2 - \frac{\ell}{4}\left(\frac{r}{\ell}\right)^2 \cos 2\,\omega\,t + \cdots \tag{1}$$

Equation of motion:

$$m\,\ddot{x} + k\,(x - y) \pm \mu\,N = 0$$

or

$$m\,\ddot{x} + k\,x \pm \mu\,N = k\,y$$

$$= \left\{k\,r + \frac{k\,\ell}{4}\left(\frac{r}{\ell}\right)^2\right\} - k\,r\cos\omega\,t - \frac{k\,\ell}{4}\left(\frac{r}{\ell}\right)^2 \cos 2\,\omega\,t + \cdots \tag{2}$$

For given numerical data, Eq. (2) becomes:

$$\ddot{x} + 100\, x \pm 0.981 = 100\, y$$

$$= \left\{100\,(0.1) + \frac{100\,(1)}{4}\left(\frac{0.1}{1}\right)^2\right\} - (100)\,(0.1)\cos 100\,t - \frac{100\,(1)}{4}\left(\frac{0.1}{1}\right)^2 \cos 200\,t - \cdots$$

$$= 10.25 - 10\cos 100\,t - 0.25\cos 200\,t - \cdots \qquad (3)$$

Using the definition of equivalent damping constant, c_{eq}, the solution of Eq. (3) can be found by superposition.

$$c_{eq} = \frac{4\,\mu\,N}{\pi\,\Omega\,X} = \frac{4\,\mu\,m\,g}{\pi\,\Omega\,X} \qquad (4)$$

where Ω is the frequency of the harmonic force and X is the amplitude of the mass.

Steady state solution due to constant term, F_0, on the r.h.s. of Eq. (3) (from Example 4.9):

$$x(t) = \frac{F_0}{k} = \left\{k\,r + \frac{k\,\ell}{4}\left(\frac{r}{\ell}\right)^2\right\} = r + \frac{\ell}{4}\left(\frac{r}{\ell}\right)^2 \qquad (5)$$

Steady state solution due to harmonic term, $F_0 \cos \Omega\, t$, on the r.h.s. of Eq. (3) (from Eqs. (3.89), (3.93) and (3.96)):

$$x(t) = X\cos(\Omega\,t - \phi) \qquad (6)$$

$$\text{where} \quad X = \frac{F_0}{k}\left[\frac{1 - \left(\frac{4\,\mu\,N}{\pi\,F_0}\right)^2}{\left(1 - \frac{\Omega^2}{\omega_n^2}\right)^2}\right]^{\frac{1}{2}} \quad \text{and} \quad \phi = \tan^{-1}\left\{\frac{\pm\frac{4\,\mu\,N}{\pi\,F_0}}{\left[1 - \left(\frac{4\,\mu\,N}{\pi\,F_0}\right)^2\right]^{\frac{1}{2}}}\right\} \qquad (7)$$

$N = m\,g = 1\,(9.81) = 9.81$ Newtons, $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{100}{1}} = 10$ rad/sec, and + sign is to be used in Eq. (7) for $\Omega < \omega_n$ and - sign for $\Omega > \omega_n$. Equations (5) and (6) can be superposed suitably to find the complete steady state solution of Eq. (3).

4.10 Base motion can be represented by Fourier series as (see solution of Problem 4.8 or Example 1.19):

$$y(t) = \frac{Y}{\pi}\left\{\frac{\pi}{2} - \left(\sin\omega\,t + \frac{1}{2}\sin 2\,\omega\,t + \frac{1}{3}\sin 3\,\omega\,t + \cdots\right)\right\} \qquad (1)$$

Equation of motion of mass:

$$m\,\ddot{x} + k\,((x - y) \pm \mu\,N = 0$$

or

$$m\ddot{x} + k\,x \pm \mu N = k\,y$$

$$= \frac{kY}{\pi}\left[\frac{\pi}{2} - \left\{\sin\omega t + \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t + \cdots\right\}\right] \quad (2)$$

Using the definition of equivalent viscous damping constant:

$$c_{eq} = \frac{4\mu N}{\pi \Omega X} = \frac{4\mu m g}{\pi \Omega X} \quad (3)$$

where Ω is the frequency of the harmonic force and X is the amplitude of the mass, the solution of Eq. (2) can be determined using the superposition principle.

Steady state solution due to constant term, F_0, on the r.h.s. of Eq. (2) (from Example 4.9):

$$x(t) = \frac{F_0}{k} = \frac{kY}{2k} = \frac{Y}{2} \quad (4)$$

Steady state solution due to harmonic term, $F_0 \sin \Omega t$, on the r.h.s. of Eq. (2) (from Eqs. (3.89), (3.93) and (3.96)):

$$x(t) = X \sin(\Omega t - \phi)$$

$$\text{where} \quad X = \frac{F_0}{k}\left[\frac{1 - \left(\frac{4\mu N}{\pi F_0}\right)^2}{\left(1 - \frac{\Omega^2}{\omega_n^2}\right)^2}\right]^{\frac{1}{2}} \quad \text{and} \quad \phi = \tan^{-1}\left\{\frac{\pm \frac{4\mu N}{\pi F_0}}{\left[1 - \left(\frac{4\mu N}{\pi F_0}\right)^2\right]^{\frac{1}{2}}}\right\} \quad (6)$$

and + sign is to be used in Eq. (6) for $\Omega < \omega_n$ and - sign for $\Omega > \omega_n$. Equations (4) and (5) can be superposed suitably to find the complete steady state solution of Eq. (2).

4.11 Base excitation: $y(t) = Y \cos \omega t$ (E.1)
with $Y = 0.05$ and $\omega = 5$

Equation of motion of the system:

$$m\ddot{x} + c\dot{x} + kx = ky + c\dot{y} = kY\cos\omega t - c\omega Y \sin\omega t \quad (E.2)$$

Equation (E.2) is similar to Eq. (4.8) with

$a_0 = 0$, $a_1 = kY$, $b_1 = -c\omega Y$ and $a_i = b_i = 0$; $i = 2, 3, \cdots$

steady state response of the system is given by

Eq. (E.9) of Example 4.9:

$$x_p(t) = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \left\{ \frac{a_1}{k} \cos(\omega t - \phi_1) + \frac{b_1}{k} \sin(\omega t - \phi_1) \right\}$$

Here $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4000}{10}} = 20$ rad/s

$r = \frac{\omega}{\omega_n} = \frac{5}{20} = 0.25$, $\quad \zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{k\,m}} = \frac{20}{2\sqrt{4000\,(10)}} = 0.05$

$\omega_d = \sqrt{1-\zeta^2}\,\omega_n = 19.975$ rad/s

$a_1 = k\,Y = 4000\,(0.05) = 200$, $\quad b_1 = -c\,\omega\,Y = -20\,(5)(0.05) = -5$

$$\phi_1 = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}\left(\frac{2\,(0.05)(0.25)}{1-0.25^2}\right) = 0.02666 \text{ rad}$$

$$\sqrt{(1-r^2)^2 + (2\zeta r)^2} = \sqrt{(1-0.25^2)^2 + (2 * 0.05 * 0.25)^2} = 0.937833$$

Solution of the homogeneous equation is given by Eq. (2.70):

$$x_h(t) = X_0\, e^{-\zeta \omega_n t} \cos(\omega_d t - \phi_0) = X_0\, e^{-t} \cos(19.975\, t - \phi_0) \qquad (E.3)$$

Where X_0 and ϕ_0 are unknown constants.

Total solution:

$$x(t) = x_h(t) + x_p(t) = X_0\, e^{-t} \cos(19.975\, t - \phi_0) + \frac{1}{0.937833}\left\{ \frac{200}{4000} \cos(5t - 0.02666) - \frac{5}{4000} \sin(5t - 0.02666) \right\}$$

$$= X_0\, e^{-t} \cos(19.975\, t - \phi_0) + 0.053314 \cos(5t - 0.02666) - 0.001333 \sin(5t - 0.02666) \qquad (E.4)$$

where X_0 and ϕ_0 are to be found from the initial conditions. Differentiation of Eq. (E.4) gives the velocity as:

$$\dot{x}(t) = -X_0\, e^{-t} \cos(19.975\, t - \phi_0) - X_0\,(19.975)\, e^{-t} \sin(19.975\, t - \phi_0)$$

$$- 0.26657 \sin(5t - 0.02666)$$

$$- 0.006665 \cos(5t - 0.02666) \qquad (E.5)$$

Using the initial conditions, we obtain

$$x_0 = x(t=0) = 0.1 = X_0 \cos\phi_0 + 0.053314 \cos(0.02666) + 0.001333 \sin(0.02666)$$

or

$$X_0 \cos\phi_0 = 0.04666947 \qquad (E.6)$$

and

$$\dot{x}_0 = \dot{x}(t=0) = 1.0 = -X_0 \cos\phi_0 + 19.975\, X_0 \sin\phi_0 + 0.26657 \sin(0.02666) - 0.006665 \cos(0.02666)$$

or

$$X_0 \sin\phi_0 = 0.05237678 \qquad (E.7)$$

Solution of Eqs. (E.6) and (E.7) gives

$$X_0 = \left\{ (X_0 \cos\phi_0)^2 + (X_0 \sin\phi_0)^2 \right\}^{\frac{1}{2}} = 0.07015247 \qquad (E.8)$$

$$\phi_0 = \tan^{-1}\left(\frac{X_0 \sin\phi_0}{X_0 \cos\phi_0} \right) = 0.84295916 \text{ rad} \qquad (E.9)$$

Complete solution is given by Eq. (E.4):

$$x(t) = 0.07015247\, e^{-t} \cos(19.975\, t - 0.84295916)$$

$$+ 0.053314 \cos(5t - 0.02666)$$

$$- 0.001333 \sin(5t - 0.02666) \qquad (E.10)$$

4.13 From solution of problem 1.117,

$$F(t) = 9.9584 - 20.1587 \cos 10.472\,t + 23.5253 \sin 10.472\,t$$
$$+ 3.3099 \cos 20.944\,t + 12.2646 \sin 20.944\,t$$
$$+ 3.7719 \cos 31.416\,t - 0.4064 \sin 31.416\,t \qquad (E_1)$$

$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{15000}{1}} = 122.4745$ rad/sec ; $\quad \zeta = 0.1$; $\quad r = 0.0855$

The steady state solution is given by Eq. (E.9) of Example 4.9:

$$x_p(t) = \frac{9.9584}{k} - \frac{20.1587}{k} \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \cos(10.472t - \phi_1)$$

$$+ \frac{23.5253}{k} \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \sin(10.472t - \phi_1)$$

$$+ \frac{3.3099}{k} \frac{1}{\sqrt{(1-4r^2)^2 + (4\zeta r)^2}} \cos(20.944t - \phi_2)$$

$$+ \frac{12.2646}{k} \frac{1}{\sqrt{(1-4r^2)^2 + (4\zeta r)^2}} \sin(20.944t - \phi_2)$$

$$+ \frac{3.7719}{k} \frac{1}{\sqrt{(1-9r^2)^2 + (6\zeta r)^2}} \cos(31.416t - \phi_3)$$

$$- \frac{0.4064}{k} \frac{1}{\sqrt{(1-9r^2)^2 + (6\zeta r)^2}} \sin(31.416t - \phi_3) \qquad (E_2)$$

where $\phi_1 = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}(0.017226) = 0.0172$ rad

$\phi_2 = \tan^{-1}\left(\frac{4\zeta r}{1-4r^2}\right) = \tan^{-1}(0.035229) = 0.0352$ rad

$\phi_3 = \tan^{-1}\left(\frac{6\zeta r}{1-9r^2}\right) = \tan^{-1}(0.054913) = 0.0549$ rad

Noting that $2\zeta r = 2(0.1)(0.0855) = 0.0171$ and $(1-r^2) = 0.9927$, Eq. (E_2) can be rewritten as

$$x_p(t) = [6.6389 - 13.7821 \cos(10.472t - 0.0172)$$
$$+ 15.7965 \sin(10.472t - 0.0172) + 2.2715 \cos(20.944t - 0.0352)$$
$$+ 8.4168 \sin(20.944t - 0.0352) + 2.6876 \cos(31.416t - 0.0549)$$
$$- 0.2896 \sin(31.416t - 0.0549)] \times 10^{-4} \text{ m.}$$

4.14 From problem 1.116,

$$F(t) = 1137.5 - 414.9436 \cos 523.6t + 150.3139 \sin 523.6t$$
$$+ 28.6058 \cos 1047.2t - 146.1706 \sin 1047.2t$$
$$+ 35.7278 \cos 1570.8t + 55.1546 \sin 1570.8t \qquad (E_1)$$

$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{8000}{0.5}} = 126.4911$ rad/sec ; $\zeta = 0.06$

$r = \frac{\omega}{\omega_n} = \frac{523.6}{126.4911} = 4.1394$; $r^2 = 17.1348$; $1-r^2 = -16.1348$

$2\zeta r = 2(0.06)(4.1394) = 0.4967$

$$x_p(t) = \frac{1137.5}{k} - \frac{414.9436}{k} \frac{\cos(523.6\,t - \phi_1)}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

$$+ \frac{150.3139}{k} \frac{\sin(523.6\,t - \phi_1)}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} + \frac{28.6058}{k} \frac{\cos(1047.2\,t - \phi_2)}{\sqrt{(1-4r^2)^2 + (4\zeta r)^2}}$$

$$- \frac{146.1706}{k} \frac{\sin(1047.2\,t - \phi_2)}{\sqrt{(1-4r^2)^2 + (4\zeta r)^2}} + \frac{35.7278}{k} \frac{\cos(1570.8\,t - \phi_3)}{\sqrt{(1-9r^2)^2 + (6\zeta r)^2}}$$

$$+ \frac{55.1546}{k} \frac{\sin(1570.8\,t - \phi_3)}{\sqrt{(1-9r^2)^2 + (6\zeta r)^2}} \qquad (E_2)$$

Since $\phi_1 = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}(-0.0308) = -0.0308$

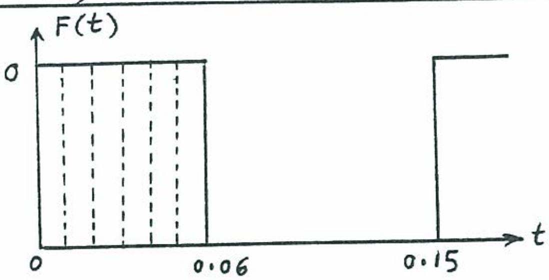
$$\phi_2 = \tan^{-1}\left(\frac{4\zeta r}{1-4r^2}\right) = \tan^{-1}(-0.0147) = -0.0147$$

$$\phi_3 = \tan^{-1}\left(\frac{6\zeta r}{1-9r^2}\right) = \tan^{-1}(-0.00973) = -0.00973$$

the steady-state solution, Eq. (E_2), can be expressed as

$$x_p(t) = 0.1422 - 0.003211 \cos(523.6\,t + 0.0308)$$
$$+ 0.001163 \sin(523.6t + 0.0308) + 5.2921 \times 10^{-5} \cos(1047.2t + 0.0147)$$
$$- 2.7042 \times 10^{-4} \sin(1047.2t + 0.0147) + 2.9145 \times 10^{-5} \cos(1570.8\,t + 0.00973)$$
$$+ 4.4992 \times 10^{-5} \sin(1570.8\,t + 0.00973) \quad m$$

4.15 $\omega_n = \sqrt{\frac{5 \times 10^6}{10 \times 10^3}} = 22.3607 \ \frac{rad}{sec}$



$\tau = 0.15$ sec

$\omega = \frac{2\pi}{\tau} = 41.888$ rad/sec

$r = \frac{\omega}{\omega_n} = 1.8733$, $r^2 = 3.5093$

$\zeta = 0$

Following data is used in Program 1.F to find Fourier coefficients in the expansion of $F(t)$:

t, sec	0.01	0.02	0.03	0.04	0.05	0.06	0.07	...	0.15
F(t), kN	400	400	400	400	400	400	0	...	0

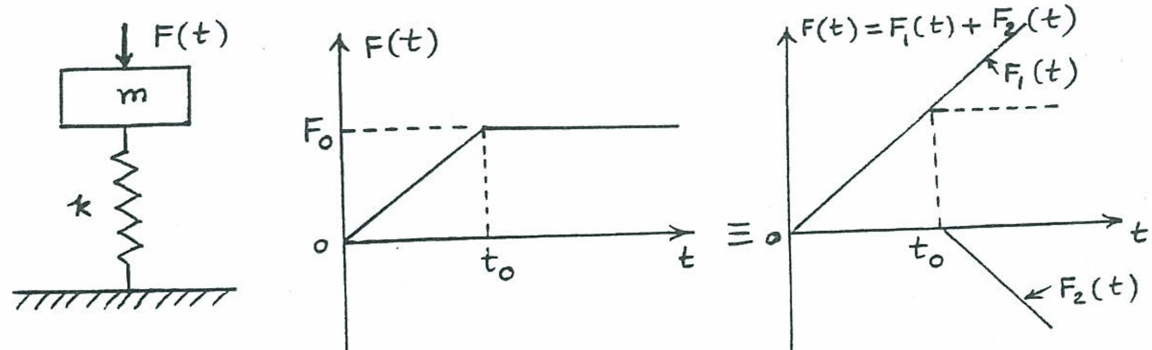
Result is:

$$F(t) = 160.0 + 25.5002 \cos 41.888t + 242.6276 \sin 41.888t$$
$$-75.3884 \cos 83.776t + 16.0237 \sin 83.776t$$
$$+16.4806 \cos 125.664t + 50.7237 \sin 125.664t$$
$$-62.3538 \cos 167.552t + 27.7604 \sin 167.552t$$
$$+ \cdots \quad kN$$

Since $\zeta = 0$ and all $\phi_j = 0$, $j = 1, 2, \cdots$, the steady-state response of the water tank, Eq. (4.13), becomes

$$x_p(t) = 0.032 + 2.0325 \times 10^{-3} \cos 41.888t$$
$$+ 19.3383 \times 10^{-3} \sin 41.888t - 1.1566 \times 10^{-3} \cos 83.776t$$
$$+ 0.2458 \times 10^{-3} \sin 83.776t + 0.1078 \times 10^{-3} \cos 125.664t$$
$$+ 0.3317 \times 10^{-3} \sin 125.664t - 0.2334 \times 10^{-3} \cos 167.552t$$
$$+ 0.1007 \times 10^{-3} \sin 167.552t + \cdots \quad m$$

4.16



Forcing function can be considered as the sum of two ramp functions, $F_1(t) = \dfrac{F_0\, t}{t_0}$ and $F_2(t) = -\dfrac{F_0 (t - t_0)}{t_0}$.

Response of the casting (undamped spring-mass system) to F_1 is given by

$$x_1(t) = \frac{F_0}{k}\left(\frac{t}{t_0} - \frac{\sin \omega_n t}{\omega_n t_0}\right) \quad \text{for } t \geq 0 \qquad (E_1)$$

Response due to F_2 can be obtained from Eq. (E_1) by replacing t by $t - t_0$ and F_0 by $-F_0$:

$$x_2(t) = -\frac{F_0}{k}\left\{\frac{t - t_0}{t_0} - \frac{\sin \omega_n (t - t_0)}{\omega_n t_0}\right\} \quad \text{for } t \geq t_0 \qquad (E_2)$$

Total response of the casting is given by

$$x(t) = x_1(t) + x_2(t) = \frac{F_0}{k}\left\{1 + \frac{\sin \omega_n (t - t_0) - \sin \omega_n t}{\omega_n t_0}\right\} \quad \text{for } t \geq t_0 \qquad (E_3)$$

4.17 From Eq. (4.31),

$$x(t) = \frac{1}{m\omega_d}\int_0^t F_o\, e^{-\alpha\tau}\, e^{-\zeta\omega_n(t-\tau)} \sin \omega_d (t-\tau)\, d\tau$$

$$= \frac{F_o}{m\omega_d} e^{-\zeta\omega_n t} \int_0^t e^{-(\alpha-\zeta\omega_n)\tau} \sin \omega_d (t-\tau)\, d\tau$$

$$x(t) = \frac{F_o\, e^{-\zeta\omega_n t}}{m\,\omega_d}\int_0^t e^{-(\alpha-\zeta\omega_n)\tau} \cdot \{\sin\omega_d t \cdot \cos\omega_d\tau - \cos\omega_d t\cdot \sin\omega_d\tau\}\, d\tau$$

$$= \frac{F_o\, e^{-\zeta\omega_n t}}{m\,\omega_d} \sin\omega_d t \left[\frac{e^{-(\alpha-\zeta\omega_n)\tau}}{(\alpha-\zeta\omega_n)^2+\omega_d^2}\left\{-(\alpha-\zeta\omega_n)\cos\omega_d\tau + \omega_d \sin\omega_d\tau\right\}\right]_0^t$$

$$- \frac{F_o\, e^{-\zeta\omega_n t}}{m\,\omega_d} \cos\omega_d t \left[\frac{e^{-(\alpha-\zeta\omega_n)\tau}}{(\alpha-\zeta\omega_n)^2+\omega_d^2}\left\{-(\alpha-\zeta\omega_n)\sin\omega_d\tau - \omega_d \cos\omega_d\tau\right\}\right]_0^t$$

$$= \frac{F_o\, e^{-\zeta\omega_n t}}{m\omega_d\left[(\alpha-\zeta\omega_n)^2+\omega_d^2\right]}\left\{\omega_d\, e^{-(\alpha-\zeta\omega_n)t} + (\alpha-\zeta\omega_n)\sin\omega_d t - \omega_d\cos\omega_d t\right\}$$

$$= \frac{F_o\, e^{-\alpha t}}{m\left[(\alpha-\zeta\omega_n)^2+\omega_d^2\right]} + \frac{F_o\, e^{-\zeta\omega_n t}}{m\omega_d\left[(\alpha-\zeta\omega_n)^2+\omega_d^2\right]}\sin(\omega_d t - \phi)$$

where $\phi = \tan^{-1}\left(\dfrac{\omega_d}{\alpha-\zeta\omega_n}\right)$.

4.18 Equation of motion:

$$m\,\ddot{x} + k\,x = A\,p(t)$$

$$\text{or}\quad 10\,\ddot{x} + 1000\,x = \frac{\pi}{4}\,(0.1)^2\,(50)\,(1 - e^{-3t}) = 0.3927 - 0.3927\,e^{-3t}$$

Solution:

$$x(t) = C_1\cos\omega_n t + C_2\sin\omega_n t + \frac{0.3927}{k} - \frac{0.3927}{k + m\,(3^2)}\,e^{-3t}$$

where $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{10}} = 10$ rad/sec. Thus x(t) becomes

$$x(t) = C_1 \cos 10\,t + C_2 \sin 10\,t + 39.27\,(10^{-5}) - 36.0275\,(10^{-5})\, e^{-3t} \text{ m}$$

where C_1 and C_2 can be determined from the initial conditions. As $t \to \infty$, $e^{-3t} \to 0$ and the steady state response becomes

$$x(t) = C_1 \cos 10\,t + C_2 \sin 10\,t + 39.27\,(10^{-5}) \text{ m}$$

4.19 For $0 \le t \le \frac{\pi}{\omega}$: Equation of motion: $m\ddot{x} + kx = \frac{F_0}{2} - \frac{F_0}{2}\cos\omega t$

$$x(t) = A\cos\omega_n t + B\sin\omega_n t + \frac{F_0}{2k} - \frac{F_0}{2k\left(1-\frac{\omega^2}{\omega_n^2}\right)}\cos\omega t$$

$$x(0)=0 \quad \therefore \quad A + \frac{F_0}{2k} - \frac{F_0}{2k\left(1-\frac{\omega^2}{\omega_n^2}\right)}, \quad A = \frac{F_0\left(\frac{\omega}{\omega_n}\right)^2}{2k\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}}$$

$$\dot{x}(0)=0 \quad \therefore \quad B = 0$$

$$x(t) = \frac{F_0}{2k\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}}\left[1-\cos\omega t - \left(\frac{\omega}{\omega_n}\right)^2(1-\cos\omega_n t)\right]$$

At $t = \frac{\pi}{\omega}$, $$x\left(\frac{\pi}{\omega}\right) = \frac{F_0}{2k\left(1-\frac{\omega^2}{\omega_n^2}\right)}\left[2 - \frac{\omega^2}{\omega_n^2}\left(1-\cos\frac{\omega_n\pi}{\omega}\right)\right]$$

For $t > \frac{\pi}{\omega}$: Eq. (4.31) gives $$x(t) = \frac{1}{m\omega_n}\int_0^t F(\tau)\sin\omega_n(t-\tau)\,d\tau$$

$$x(t) = \frac{1}{m\omega_n}\int_0^{\pi/\omega} F(\tau)\cdot\sin\omega_n(t-\tau)\,d\tau + \frac{1}{m\omega_n}\int_{\pi/\omega}^{t} F(\tau)\sin\omega_n(t-\tau)\,d\tau$$

$$= x\left(t=\frac{\pi}{\omega}\right) + \frac{F_0}{m\omega_n}\int_{\pi/\omega}^{t}\sin\omega_n(t-\tau)\,d\tau$$

$$= x\left(t=\frac{\pi}{\omega}\right) + \frac{F_0}{m\omega_n^2}\Big[\cos\omega_n(t-\tau)\Big]_{\tau=\pi/\omega}^{t}$$

$$= \frac{F_0}{2k\left(1-\frac{\omega^2}{\omega_n^2}\right)}\left[2-\frac{\omega^2}{\omega_n^2}\left(1-\cos\frac{\omega_n\pi}{\omega}\right)\right] + \frac{F_0}{k}\left[1-\cos\omega_n\left(t-\frac{\pi}{\omega}\right)\right]$$

4.20 For an undamped system Eq. (4.31) gives

$$x(t) = \frac{1}{m\omega_n}\int_0^t F(\tau)\sin\omega_n(t-\tau)\,d\tau$$

$$F(\tau) = \begin{cases} F_0 & \text{for } 0 \le \tau \le t_0 \\ 0 & \text{for } \tau > t_0 \end{cases}$$

For $0 \le t \le t_0$: $\quad x(t) = \dfrac{F_0}{m\omega_n} \displaystyle\int_0^t \sin \omega_n (t-\tau)\, d\tau = \dfrac{F_0}{m\omega_n^2}\left(1 - \cos \omega_n t\right)$

$$= \frac{F_0}{k}\left(1 - \cos \omega_n t\right)$$

For $t > t_0$: $\quad x(t) = \dfrac{F_0}{m\omega_n} \displaystyle\int_0^{t_0} \sin \omega_n (t-\tau)\, d\tau = \dfrac{F_0}{m\omega_n}\left[\dfrac{1}{\omega_n} \cos \omega_n (t-\tau)\right]_{\tau=0}^{t_0}$

$$= \frac{F_0}{k}\left[\cos \omega_n (t - t_0) - \cos \omega_n t\right]$$

4.21 $F(\tau) = \begin{cases} F_0\left(\dfrac{\tau}{t_0}\right) & \text{for } 0 \le \tau \le t_0 \\ 0 & \text{for } \tau > t_0 \end{cases}$

For $0 \le t \le t_0$: $\quad x(t) = \dfrac{F_0}{m\omega_n t_0} \displaystyle\int_0^t \tau \sin \omega_n (t-\tau)\, d\tau$

i.e. $\quad x(t) = \dfrac{F_0}{m\omega_n t_0}\left[\displaystyle\int_0^t (t-\tau) \sin \omega_n (t-\tau)\,(-d\tau) - t \int_0^t \sin \omega_n (t-\tau)\,(-d\tau)\right]$

$$= \frac{F_0}{m\omega_n t_0}\left[\frac{1}{\omega_n^2} \sin \omega_n (t-\tau) - \frac{(t-\tau)}{\omega_n} \cos \omega_n (t-\tau)\right]_{\tau=0}^{t}$$

$$+ \frac{F_0\, t}{m\omega_n t_0}\left[\frac{\cos \omega_n (t-\tau)}{\omega_n}\right]_{\tau=0}^{t}$$

$$= \frac{F_0}{k t_0}\left(t - \frac{1}{\omega_n} \sin \omega_n t\right)$$

For $t > t_0$:

$$x(t) = \frac{F_0}{m\omega_n t_0} \int_0^{t_0} \tau \sin \omega_n (t-\tau)\, d\tau$$

$$= \frac{F_0}{m\omega_n t_0}\left[\frac{1}{\omega_n^2} \sin \omega_n (t-\tau) - \frac{(t-\tau)}{\omega_n} \cos \omega_n (t-\tau)\right]_{\tau=0}^{t}$$

$$+ \frac{F_0\, t}{m\omega_n t_0}\left[\frac{\cos \omega_n (t-\tau)}{\omega_n}\right]_{\tau=0}^{t}$$

$$= \frac{F_0}{k t_0}\left[\frac{1}{\omega_n} \sin \omega_n (t - t_0) + t_0 \cos \omega_n (t - t_0) - \frac{1}{\omega_n} \sin \omega_n t\right]$$

4.22 $F(t) = \begin{cases} F_0\left(1 - \cos \dfrac{\pi t}{2 t_0}\right) & ;\ 0 \le t \le t_0 \\ 0 & ;\ t > t_0 \end{cases} \qquad (E_1)$

For an undamped system, Eq. (4.31) gives

$$x(t) = \frac{1}{m\omega_n} \int_0^t F(\tau) \sin \omega_n (t-\tau)\, d\tau$$

For $0 \le t \le t_0$:

$$x(t) = \frac{F_0}{m\omega_n} \int_0^t \left(1 - \cos\frac{\pi\tau}{2t_0}\right) \sin\omega_n(t-\tau)\, d\tau \qquad (E_2)$$

Noting that

$$\int_0^t \sin(\omega_n t - \omega_n \tau)\, d\tau = \left[\frac{1}{\omega_n}\cos(\omega_n t - \omega_n\tau)\right]_0^t = \frac{1}{\omega_n}(1 - \cos\omega_n t)$$

and

$$\int_0^t \cos\frac{\pi\tau}{2t_0}\left(\sin\omega_n t \cos\omega_n\tau - \cos\omega_n t \sin\omega_n\tau\right) d\tau$$

$$= \sin\omega_n t \left[\frac{\sin\left(\frac{\pi}{2t_0} - \omega_n\right)\tau}{2\left(\frac{\pi}{2t_0} - \omega_n\right)} + \frac{\sin\left(\frac{\pi}{2t_0} + \omega_n\right)\tau}{2\left(\frac{\pi}{2t_0} + \omega_n\right)}\right]_0^t$$

$$-\cos\omega_n t \left[-\frac{\cos\left(\omega_n - \frac{\pi}{2t_0}\right)\tau}{2\left(\omega_n - \frac{\pi}{2t_0}\right)} - \frac{\cos\left(\omega_n + \frac{\pi}{2t_0}\right)\tau}{2\left(\omega_n + \frac{\pi}{2t_0}\right)}\right]_0^t$$

$$= \frac{-1}{2\left(\frac{\pi}{2t_0} - \omega_n\right)}\left[\cos\frac{\pi t}{2t_0} - \cos\omega_n t\right] + \frac{1}{2\left(\frac{\pi}{2t_0} + \omega_n\right)}\left[\cos\frac{\pi t}{2t_0} - \cos\omega_n t\right],$$

Eq. (E_2) can be simplified as

$$x(t) = \frac{F_0}{k}(1 - \cos\omega_n t) + \frac{(F_0/m)}{\left\{\left(\frac{\pi}{2t_0}\right)^2 - \omega_n^2\right\}}\left[\cos\frac{\pi t}{2t_0} - \cos\omega_n t\right] \qquad (E_3)$$

For $t > t_0$:

$$x(t) = \frac{F_0}{m\omega_n}\int_0^{t_0}\left(1 - \cos\frac{\pi\tau}{2t_0}\right)\sin\omega_n(t-\tau)\, d\tau \qquad (E_4)$$

$$= \frac{F_0}{m\omega_n}\left[\frac{1}{\omega_n}\cos\omega_n(t-t_0) - \frac{1}{\omega_n}\cos\omega_n t\right] - \frac{F_0}{m\omega_n}\left\{\sin\omega_n t \, *\right.$$

$$\left[\frac{\sin\left(\frac{\pi}{2t_0} - \omega_n\right)t_0}{2\left(\frac{\pi}{2t_0} - \omega_n\right)} + \frac{\sin\left(\frac{\pi}{2t_0} + \omega_n\right)t_0}{2\left(\frac{\pi}{2t_0} + \omega_n\right)}\right] + \cos\omega_n t \, *$$

$$\left.\left[\frac{\cos\left(\frac{\pi}{2t_0} - \omega_n\right)t_0}{2\left(\omega_n - \frac{\pi}{2t_0}\right)} + \frac{\cos\left(\frac{\pi}{2t_0} + \omega_n\right)t_0}{2\left(\omega_n + \frac{\pi}{2t_0}\right)} - \frac{1}{2\left(\omega_n - \frac{\pi}{2t_0}\right)} - \frac{1}{2\left(\omega_n + \frac{\pi}{2t_0}\right)}\right]\right\}$$

i.e., $x(t) = \frac{F_0}{k}\left[\cos\omega_n(t - t_0) - \cos\omega_n t\right]$

$$- \frac{F_0}{2m\,\omega_n\left(\frac{\pi}{2t_0} - \omega_n\right)} \sin \omega_n (t - t_0) - \frac{F_0}{2m\,\omega_n\left(\frac{\pi}{2t_0} + \omega_n\right)} \sin \omega_n (t - t_0)$$

$$+ \frac{F_0}{2m\,\omega_n} \cos \omega_n t \cdot \left\{ - \frac{1}{\left(\frac{\pi}{2t_0} - \omega_n\right)} + \frac{1}{\left(\frac{\pi}{2t_0} + \omega_n\right)} \right\} \qquad (E_5)$$

4.23 Base displacement $= y(s) = Y \sin \frac{\pi s}{\delta}$ $\qquad (E_1)$

i.e., $\quad y(t) = Y \sin \frac{\pi t}{t_0}$ $\qquad (E_2)$

steady-state relative displacement can be found from Eq. (4.34) as

$$z(t) = -\frac{1}{\omega_d} \int_0^t \ddot{y}(\tau)\, e^{-\zeta\omega_n(t-\tau)} \sin \omega_d (t-\tau)\, d\tau \qquad (E_3)$$

where $\quad \ddot{y}(\tau) = -Y\left(\frac{\pi}{t_0}\right)^2 \sin \frac{\pi\tau}{t_0}$ $\qquad (E_4)$

$$z(t) = \frac{Y}{\omega_d}\left(\frac{\pi}{t_0}\right)^2 \int_0^t e^{-\zeta\omega_n t}\, e^{\zeta\omega_n\tau} \sin\frac{\pi\tau}{t_0} \sin\omega_d(t-\tau)\, d\tau \qquad (E_5)$$

But $\quad \sin\frac{\pi\tau}{t_0}\sin\omega_d(t-\tau) = \frac{1}{2}\cos\left(\frac{\pi\tau}{t_0} - \omega_d t + \omega_d\tau\right) - \frac{1}{2}\cos\left(\frac{\pi\tau}{t_0} + \omega_d t - \omega_d \tau\right)$

$$= \frac{1}{2}\left[\cos\left(\frac{\pi}{t_0} + \omega_d\right)\tau \cdot \cos\omega_d t + \sin\left(\frac{\pi}{t_0} + \omega_d\right)\tau \cdot \sin\omega_d t\right]$$

$$-\frac{1}{2}\left[\cos\left(\frac{\pi}{t_0} - \omega_d\right)\tau \cdot \cos\omega_d t - \sin\left(\frac{\pi}{t_0} - \omega_d\right)\tau \cdot \sin\omega_d t\right] \qquad (E_6)$$

Eqs. (E_5) and (E_6) give

$$z(t) = \frac{1}{2}\,\frac{Y}{\omega_d}\left(\frac{\pi}{t_0}\right)^2 e^{-\zeta\omega_n t}\left[\cos\omega_d t \cdot \int_0^t e^{\zeta\omega_n\tau} \cos\left(\frac{\pi}{t_0}+\omega_d\right)\tau \cdot d\tau\right.$$

$$+ \sin\omega_d t \cdot \int_0^t e^{\zeta\omega_n\tau} \cdot \sin\left(\frac{\pi}{t_0}+\omega_d\right)\tau \cdot d\tau$$

$$- \cos\omega_d t \cdot \int_0^t e^{\zeta\omega_n\tau} \cdot \cos\left(\frac{\pi}{t_0}-\omega_d\right)\tau \cdot d\tau$$

$$\left. + \sin\omega_d t \cdot \int_0^t e^{\zeta\omega_n\tau} \cdot \sin\left(\frac{\pi}{t_0}-\omega_d\right)\tau \cdot d\tau\right] \qquad (E_7)$$

Eq. (E_7) can be simplified

$$z(t) = \frac{Y}{2\omega_d}\left(\frac{\pi}{t_0}\right)^2 \frac{1}{(\zeta\omega_n)^2 + \left(\frac{\pi}{t_0}+\omega_d\right)^2}\left\{\zeta\omega_n \cos\frac{\pi t}{t_0}\right.$$

$$+ \left(\frac{\pi}{t_0}+\omega_d\right)\sin\frac{\pi t}{t_0} - \zeta\omega_n\, e^{-\zeta\omega_n t}\cos\omega_d t$$

$$\left. + \left(\frac{\pi}{t_0}+\omega_d\right) e^{-\zeta\omega_n t} \sin\omega_d t\right\}$$

$$+ \frac{Y}{2\omega_d}\left(\frac{\pi}{t_0}\right)^2 \frac{1}{(\zeta\omega_n)^2 + \left(\frac{\pi}{t_0}-\omega_d\right)^2}\left\{-\zeta\omega_n\cos\frac{\pi t}{t_0}\right.$$

$$- \left(\frac{\pi}{t_o} - \omega_d \right) \sin \frac{\pi t}{t_o} + \zeta \omega_n \, e^{-\zeta \omega_n t} \cos \omega_d t$$

$$+ \left(\frac{\pi}{t_o} - \omega_d \right) e^{-\zeta \omega_n t} \sin \omega_d t \Big\} \qquad (E_8)$$

4.24 Base displacement:

$$y(t) = \begin{cases} \dfrac{Y\,v\,t}{\delta} \;;\; 0 \le t \le t_0 = \dfrac{\delta}{v} \\ 0 \;;\; t > t_0 = \dfrac{\delta}{v} \end{cases} \qquad (1)$$

Equation of motion of vehicle:

$$m\,\ddot{x} + k\,(x - y) = 0 \qquad (2)$$

Using Eq. (1), Eq. (2) can be expressed as

$$m\,\ddot{x} + k\,x = k\,y = \begin{cases} \dfrac{k\,v\,Y\,t}{\delta} \;;\; 0 \le t \le t_0 \\ 0 \;;\; t > t_0 \end{cases} \qquad (3)$$

Steady state solution of Eq. (3) from Example 4.9:

$$x(t) = \begin{cases} \dfrac{v\,Y}{\delta\,\omega_n\,k} \left\{ \omega_n\,t - \sin \omega_n\,t \right\} \;;\; 0 \le t \le t_0 \\ 0 \;;\; t > t_0 \end{cases} \qquad (4)$$

Note that the homogeneous solution,

$$x(t) = C_1 \cos \omega_n\,t + C_2 \sin \omega_n\,t \qquad (5)$$

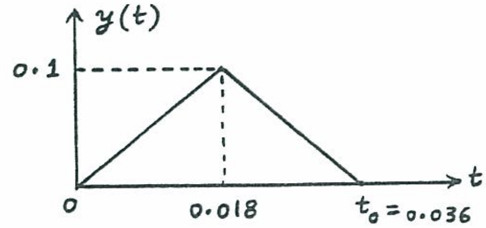
is to be added to Eq. (4) to obtain the complete solution. The constants C_1 and C_2 are to be evaluated from the initial conditions (at $t = 0$). In fact, tthe resulting complete solution is valid for all values of t, including values of $t > t_0$.

4.25 Speed of automobile = 50 km/hr

Excitation frequency $= \left(\dfrac{50 \times 1000}{3600} \right) \dfrac{1}{0.5} = 27.7778$ Hz

Natural frequency $= f_n = 1.0$ Hz $\Rightarrow \omega_n = 2\pi$ rad/sec

$$t_o = \frac{0.5 \times 3600}{50 \times 1000} = 0.036 \text{ sec}$$



$$y(t) = \begin{cases} \dfrac{0.2\,t}{t_o} \;;\; 0 \le t \le t_o/2 \\ -\dfrac{0.2\,t}{t_o} + 0.2 \;;\; \dfrac{t_o}{2} \le t \le t_o \\ 0 \;;\; t > t_o \end{cases} \quad \ldots (E_1)$$

Equation of motion (for undamped case):

$$m\ddot{x} + k(x-y) = 0 \qquad \text{or} \qquad m\ddot{x} + kx = ky = F(t) \qquad (E_2)$$

$$\text{where} \qquad F(t) = k\,y(t) \qquad (E_3)$$

Solution of Eq. (E_2) is: $$x(t) = \frac{1}{m\omega_n}\int_0^t F(\tau)\sin\omega_n(t-\tau)\,d\tau \qquad (E_4)$$

For $0 \le t \le \frac{t_0}{2}$:

$$x(t) = \frac{k}{m\omega_n}\int_0^t \left(\frac{0.2}{t_0}\right)\tau\sin\omega_n(t-\tau)\,d\tau \qquad (E_5)$$

Since $$\int_0^t \tau\sin\omega_n(t-\tau)\,d\tau = \left(\frac{t}{\omega_n} - \frac{1}{\omega_n^2}\sin\omega_n t\right),$$

Eq. (E_5) becomes

$$x(t) = 5.5556\,(t - 0.1592\sin 6.2832\,t)\ \text{m} \quad ; 0 \le t \le 0.018 \text{ sec} \qquad (E_6)$$

For $\frac{t_0}{2} \le t \le t_0$:

$$x(t) = \frac{k}{m\omega_n}\left\{\int_0^{t_0/2}\frac{0.2\tau}{t_0}\sin\omega_n(t-\tau)\,d\tau + \int_{t_0/2}^{t}\left(-\frac{0.2\tau}{t_0}+0.2\right)\sin\omega_n(t-\tau)\,d\tau\right\}$$

But

$$\frac{0.2k}{m\omega_n t_0}\cdot\int_0^{t_0/2}\tau\cdot\sin\omega_n(t-\tau)\,d\tau = \frac{0.2k}{m\omega_n t_0}\left\{\sin\omega_n t\left[\frac{1}{\omega_n^2}\cos\omega_n\tau + \frac{\tau}{\omega_n}\sin\omega_n\tau\right]_0^{t_0/2} - \cos\omega_n t\left[\frac{1}{\omega_n^2}\sin\omega_n\tau - \frac{\tau}{\omega_n}\cos\omega_n\tau\right]_0^{t_0/2}\right\}$$

$$= 5.5556\,[0.1592\sin 6.2832(t-0.018) + 0.0180\cos 6.2832\,(t-0.018) - 0.1592\sin 6.2832\,t]\ \text{m} \qquad (E_7)$$

since $t_0 = 0.036$.

$$-\frac{0.2k}{m\omega_n t_0}\int_{t_0/2}^{t}\tau\sin\omega_n(t-\tau)\,d\tau = -\frac{0.2k}{m\omega_n t_0}\left\{\sin\omega_n t\left[\frac{1}{\omega_n^2}\cos\omega_n\tau + \frac{\tau}{\omega_n}\sin\omega_n\tau\right]_{t_0/2}^{t} - \cos\omega_n t\cdot\left[\frac{1}{\omega_n^2}\sin\omega_n\tau - \frac{\tau}{\omega_n}\cos\omega_n\tau\right]_{t_0/2}^{t}\right\}$$

$$= -5.5556\,[t - 0.1592\sin 6.2832\,(t-0.018) - 0.0180\cos 6.2832\,(t-0.018)]\ \text{m} \qquad (E_8)$$

$$\frac{0.2\,k}{m\,\omega_n}\int_{t_0/2}^{t}\sin\omega_n(t-\tau)\,d\tau = \frac{0.2\,k}{m\,\omega_n}\left[\frac{1}{\omega_n}\cos\omega_n(t-\tau)\right]_{t_0/2}^{t}$$

$$= \frac{0.2\,k}{m\,\omega_n^2}\left[1-\cos\omega_n\left(t-\frac{t_0}{2}\right)\right]$$

$$= 0.2\left[1-\cos 6.2832\,(t-0.018)\right]\ m \qquad (E_9)$$

Hence the solution can be expressed as

$$x(t) = \left[1.7689\sin 6.2832\,(t-0.018) - 0.8845\sin 6.2832\,t\right.$$
$$\left. -5.5556\,t + 0.2\right]\ m\ ; \qquad 0.018 \le t \le 0.036\ \text{sec} \qquad (E_{10})$$

For $t > t_0$:

$$x(t) = \frac{0.2\,k}{m\,\omega_n t_0}\int_0^{t_0/2}\tau\sin\omega_n(t-\tau)\,d\tau - \frac{0.2\,k}{m\,\omega_n t_0}\int_{t_0/2}^{t_0}\tau\sin\omega_n(t-\tau)\,d\tau$$
$$+ \frac{0.2\,k}{m\,\omega_n}\int_{t_0/2}^{t_0}\sin\omega_n(t-\tau)\,d\tau \qquad (E_{11})$$

The first term on the right side of (E_{11}) is given by (E_7).

Second term on the right side of (E_{11}) is

$$-\frac{0.2\,k}{m\,\omega_n t_0}\int_{t_0/2}^{t_0}\tau\sin\omega_n(t-\tau)\,d\tau = -\frac{0.2\,k}{m\,\omega_n t_0}\left\{\sin\omega_n t\cdot\left[\frac{1}{\omega_n^2}\cos\omega_n\tau\right.\right.$$
$$\left.\left. + \frac{\tau}{\omega_n}\sin\omega_n\tau\right]_{t_0/2}^{t_0} - \cos\omega_n t\cdot\left[\frac{1}{\omega_n^2}\sin\omega_n\tau - \frac{\tau}{\omega_n}\cos\omega_n\tau\right]_{t_0/2}^{t_0}\right\}$$

$$= -5.5556\left[0.1592\sin 6.2832\,(t-0.036) + 0.0360\cos 6.2832\,(t-0.036)\right.$$
$$\left. -0.1592\sin 6.2832\,(t-0.018) - 0.0180\cos 6.2832\,(t-0.018)\right]\ m \qquad (E_{12})$$

The third term on the right side of (E_{11}) is:

$$\frac{0.2\,k}{m\,\omega_n}\int_{t_0/2}^{t_0}\sin\omega_n(t-\tau)\,d\tau = \frac{0.2\,k}{m\,\omega_n}\left[\frac{1}{\omega_n}\cos\omega_n(t-\tau)\right]_{t_0/2}^{t_0}$$

$$= \frac{0.2\,k}{m\,\omega_n^2}\left[\cos\omega_n(t-t_0) - \cos\omega_n\left(t-\frac{t_0}{2}\right)\right]$$

$$= 0.2\left[\cos 6.2832\,(t-0.036) - \cos 6.2832\,(t-0.018)\right]\ m \qquad (E_{13})$$

$\therefore$ $x(t)$ is given by the sum of Eqs. (E_7), (E_{12}) and (E_{13}), which can be simplified as

$$x(t) = 1.7689 \sin 6.2832\,(t - 0.018) - 0.8845 \sin 6.2832\, t$$
$$- 0.8845 \sin 6.2832\,(t - 0.036) \quad m \; ; \quad t > 0.036 \text{ sec} \qquad (E_{14})$$

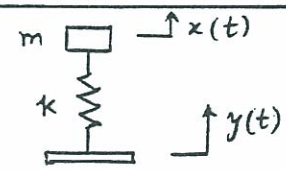
4.26 When the container strikes the floor, the velocity of the mass is given by

$$m g h = \frac{1}{2} m v^2 \quad \text{or} \quad v = \sqrt{2 g h} \qquad (E_1)$$

The displacement of the camcorder subjected to an initial velocity $\dot{x}_0 = v$ is given by Eq. (2.72), with $x_0 = 0$ and $\zeta < 1$,

$$x(t) = e^{-\zeta \omega_n t} \cdot \frac{\dot{x}_0}{\omega_n \sqrt{1 - \zeta^2}} \cdot \sin \sqrt{1 - \zeta^2}\, \omega_n t \qquad (E_2)$$

4.27 System can be modeled as a spring-mass system subjected to base motion:



$$y(t) = \begin{cases} (Y t^2 / t_0^2) \; ; & 0 \le t \le t_0 \\ 0 \; ; & t > t_0 \end{cases} \qquad \text{---} \; (E_1)$$

Relative displacement of mass, $z = x - y$, is given by Eq. (4.34):

$$z(t) = -\frac{1}{\omega_d} \int_0^t \ddot{y}(\tau)\, e^{-\zeta \omega_n (t - \tau)} \sin \omega_d (t - \tau)\, d\tau \qquad (E_2)$$

where
$$\left.\begin{aligned} y(\tau) &= (Y \tau^2 / t_0^2) \\ \ddot{y}(\tau) &= (2Y / t_0^2) \end{aligned}\right\} \; ; \quad 0 \le \tau \le t_0 \qquad (E_3)$$

and $\quad y(\tau) = 0 \quad ; \quad \tau > t_0$

For $0 \le t \le t_0$:

Since the system is undamped, $\omega_d = \omega_n$ and $\zeta = 0$, and (E_2) reduces to

$$z(t) = -\frac{2Y}{\omega_n t_0^2} \int_0^t \sin \omega_n (t - \tau)\, d\tau \qquad (E_4)$$

Here

$$\int_0^t \sin \omega_n (t - \tau)\, d\tau = \int_0^t (\sin \omega_n t \cos \omega_n \tau - \cos \omega_n t \sin \omega_n \tau)\, d\tau$$
$$= \sin \omega_n t \int_0^t \cos \omega_n \tau\, d\tau - \cos \omega_n t \int_0^t \sin \omega_n \tau\, d\tau$$

$$= \sin \omega_n t \left(\frac{1}{\omega_n} \sin \omega_n \tau \right)_0^t - \cos \omega_n t \left(- \frac{1}{\omega_n} \cos \omega_n \tau \right)_0^t$$

$$= \left(\frac{1 - \cos \omega_n t}{\omega_n} \right) \qquad (E_5)$$

Thus Eq. (E_4) gives

$$x(t) = \frac{Y\, t^2}{t_0^2} - \frac{2Y}{t_0^2\, \omega_n^2} \left(1 - \cos \omega_n t \right) \;;\quad 0 \le t \le t_0 \qquad (E_6)$$

For $t > t_0$:

Eq. (E_2) gives

$$z(t) = -\frac{1}{\omega_n} \int_0^{t_0} \frac{2Y}{t_0^2} \sin \omega_n (t - \tau)\, d\tau \qquad (E_7)$$

But

$$\int_0^{t_0} \sin \omega_n (t - \tau)\, d\tau = \frac{1}{\omega_n} \left\{ \cos \omega_n (t - t_0) - \cos \omega_n t \right\}$$

$$\therefore\; z(t) = x(t) - y(t) = -\frac{2Y}{\omega_n^2\, t_0^2} \left\{ \cos \omega_n (t - t_0) - \cos \omega_n t \right\} \;;\quad t > t_0 \qquad (E_8)$$

4.28 $$I_1 = \int_0^t (t - \tau)\, e^{-\zeta \omega_n (t - \tau)} \sin \omega_d (t - \tau) (-d\tau)$$

$$= \left[\frac{(t-\tau)\, e^{-\zeta\omega_n (t-\tau)}}{(\zeta\omega_n)^2 + \omega_d^2} \left\{ -\zeta\omega_n \sin \omega_d (t-\tau) - \omega_d \cos \omega_d (t-\tau) \right\} - \frac{e^{-\zeta\omega_n (t-\tau)}}{(\zeta\omega_n)^2 + \omega_d^2} \left\{ \left(\zeta^2 \omega_n^2 - \omega_d^2 \right) \sin \omega_d (t-\tau) + 2\zeta\omega_n\omega_d \cos \omega_d (t-\tau) \right\} \right]_{\tau=0}^t$$

$$= -\frac{1}{\omega_n^4} (2\zeta\omega_n\omega_d) + \frac{t\, e^{-\zeta\omega_n t}}{\omega_n^2} \left(\zeta\omega_n \sin \omega_d t + \omega_d \cos \omega_d t \right) + \frac{e^{-\zeta\omega_n t}}{\omega_n^4} \left\{ \omega_n^2 (2\zeta^2 - 1) \sin \omega_d t + 2\zeta\omega_n\omega_d \cos \omega_d t \right\}$$

$$I_2 = \int_0^t e^{-\zeta\omega_n (t-\tau)} \sin \omega_d (t-\tau) (-d\tau)$$

$$= \left[\frac{e^{-\zeta\omega_n (t-\tau)}}{\zeta^2\omega_n^2 + \omega_d^2} \left\{ -\zeta\omega_n \sin \omega_d (t-\tau) - \omega_d \cos \omega_d (t-\tau) \right\} \right]_{\tau=0}^t$$

$$= -\frac{\omega_d}{\omega_n^2} + \frac{e^{-\zeta\omega_n t}}{\omega_n^2} \left(\zeta\omega_n \sin \omega_d t + \omega_d \cos \omega_d t \right)$$

$$x(t) = \frac{\delta F}{m\,\omega_d} \cdot I_1 - \frac{\delta F \cdot t}{m\,\omega_d} \cdot I_2$$

$$= \frac{\delta F}{k}\left[t - \frac{2\zeta}{\omega_n} + e^{-\zeta \omega_n t} \left\{ \frac{2\zeta}{\omega_n} \cos \omega_d t - \left(\frac{\omega_d^2 - \omega_n^2}{\omega_n^2 \omega_d} \right) \sin \omega_d t \right\} \right]$$

4.29 (1) $m\ddot{x} + c\dot{x} + kx = F(t)$

where $m(t) = M - m_0 t = (2000 - 10t)$ kg

and $F(t) = m_0 v = 10\,(2000) = 20000$ N

(2) With $m = M - \frac{1}{2} m_0 t_0 = 2000 - \frac{1}{2}(10)(100) = 1500$ kg, equation of motion becomes

$$1500\,\ddot{x} + 0.1 \times 10^6\,\dot{x} + 7.5 \times 10^6\,x = 20\,000 = \text{constant}$$

Maximum steady state displacement is

$$x_p(t) = \frac{F}{k} = \frac{20000}{7.5 \times 10^6} = 0.002667 \text{ m}$$

4.30 From Eq. (4.30), the response to unit step function can be obtained by setting $F(\tau) = 1$ as

$$h(t) = \int_0^t g(t-\tau)\, d\tau \qquad \text{---- } (E_1)$$

By differentiating this equation with respect to t, we obtain

$$\frac{dh}{dt}(t) = g(t)$$

4.31 Equation (4.30) gives $x(t) = \int_0^t F(\tau) \cdot g(t-\tau) \cdot d\tau$

But $g(t-\tau) = \frac{dh}{d\tau}(t-\tau)$ from problem 4.30.

$$x(t) = \int_0^t F(\tau)\, \frac{dh}{d\tau}(t-\tau)\, d\tau$$

Integration by parts gives

$$x(t) = -F(t) \cdot h(t-\tau) \Big|_{\tau=0}^{t} + \int_0^t \frac{dF}{d\tau} \cdot h(t-\tau)\, d\tau$$

$$= -F(t)\, h(0) + F(0)\, h(t) + \int_0^t \frac{dF}{d\tau}\, h(t-\tau)\, d\tau$$

But $h(0) = 0$ from Eq. (E_1) of problem 4.30.

$$\therefore\ x(t) = F(0)\, h(t) + \int_0^t \frac{dF}{d\tau}(\tau) \cdot h(t-\tau) \cdot d\tau$$

4.32 Equation of motion for rotation about O:

$$J_0\, \ddot{\theta} + M\, \ddot{x}\, (\ell) + k_1\, a^2\, \theta + k_2\, b^2\, \theta = F_0\, \ell\, e^{-t} \qquad (1)$$

where $\ddot{x} = \ell\, \ddot{\theta}$ and

$$J_0 = \frac{1}{12}\, m\, \ell^2 + m\, (\frac{\ell}{2})^2 = \frac{1}{3}\, m\, \ell^2$$

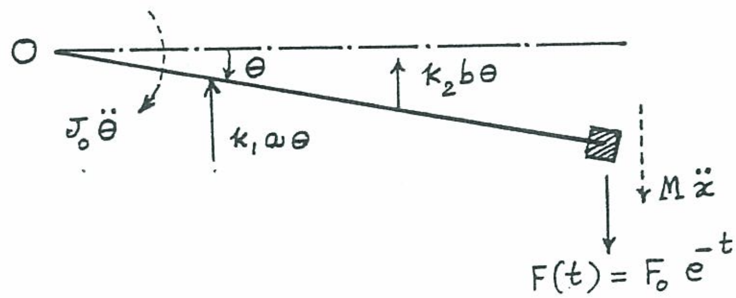
Eq. (1) can be rewritten as:

$$\left(\frac{1}{3}\ m\,\ell^2 + M\,\ell^2\right)\ddot{\theta} + (k_1\ a^2 + k_2\ b^2)\ \theta = F_0\ \ell\ e^{-t} \tag{2}$$

For given data, Eq. (2) takes the form:

$$53.3333\ \ddot{\theta} + 1562.5\ \theta = 500\ e^{-t} \tag{3}$$

Noting that the system is undamped with $\omega_n = \sqrt{\dfrac{1562.5}{53.3333}} = 5.4127$ rad/sec and



the forcing term as $500\ e^{-t}$, the convolution integral, Eq. (4.31), can be used to find the steady state response as:

$$\theta(t) = \frac{1}{(53.3333)\ (5.4127)} \int_0^t 500\ e^{-\tau}\ \sin 5.4127\ (t-\tau)\ d\tau$$

$$= 1.7320 \int_0^t e^{-t}\ e^{(t-\tau)}\ \sin 5.4127\ (t-\tau)\ d\tau$$

$$= -\ 1.7320\ e^{-t} \int_0^t e^{(t-\tau)}\ \sin 5.4127\ (t-\tau)\ (-\ d\tau) \tag{4}$$

Using the formula:

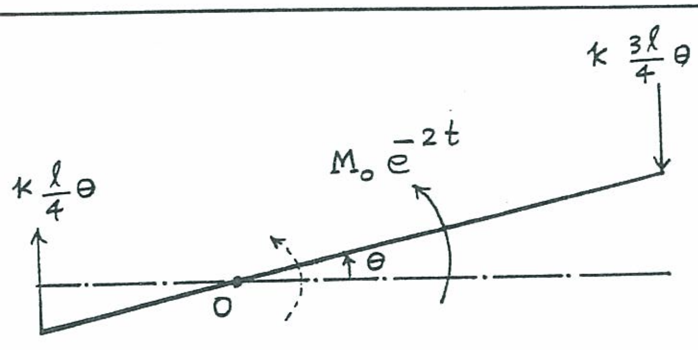
$$\int e^{a\,x}\ \sin b\ x\ dx = \frac{1}{a^2 + b^2}\ e^{a\,x}\left[a\ \sin b\ x - b\ \cos b\ x\right] \tag{5}$$

Eq. (4) can be expressed as

$$\theta(t) = -\ 1.7320\ e^{-t}\left[\frac{e^{(t-\tau)}}{1^2 + 5.4127^2}\left\{\sin 5.4127\ (t-\tau) - 5.4127\ \cos 5.4127\ (t-\tau)\right\}\right]_{\tau=0}^{\tau=t}$$

$$= 0.3094\ e^{-t} + 0.05717\ \sin 5.4127\ t - 0.3094\ \cos 5.4127\ t \text{ radian}$$

4.33



$$J_0 = \frac{1}{12} m \ell^2 + m \left(\frac{\ell}{4}\right)^2 = \frac{7}{48} m \ell^2 = \frac{7}{48} (10)(1^2) = 1.4583 \text{ kg—m}^2$$

Equation of motion for rotation about O:

$$J_0 \ddot{\theta} + k \frac{\ell^2}{16} \theta + k \frac{9}{16} \ell^2 \theta = M_0 e^{-2t}$$

$$\text{or} \quad J_0 \ddot{\theta} + \frac{5}{8} k \ell^2 \theta = M_0 e^{-2t}$$

$$\text{or} \quad 1.4583 \ddot{\theta} + 3125.0 \theta = 100 e^{-2t} \tag{1}$$

Noting that the system is undamped with $\omega_n = \sqrt{\frac{3125.0}{1.4583}} = 46.2915$ rad/sec, the convolution integral, Eq. (4.31), can be used to find the steady state response as:

$$\theta(t) = \frac{1}{(1.4583)(46.2915)} \int_0^t (100 e^{-2\tau}) \sin 46.2915 (t - \tau) d\tau$$

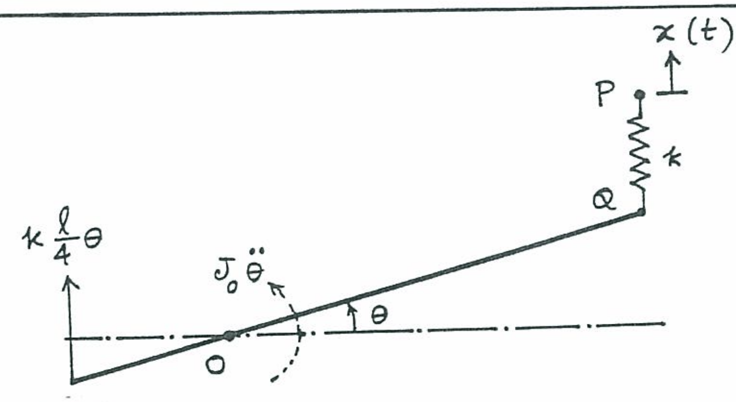
$$= -1.4813 e^{-2t} \int_0^t e^{2(t-\tau)} \sin 46.2915 (t-\tau)(-d\tau) \tag{2}$$

Using Eq. (5) in the solution of Problem 4.32, Eq. (2) can be expressed as:

$$\theta(t) = -1.4813 e^{-2t} \left[\frac{e^{2(t-\tau)}}{2^2 + 46.2915^2} \left\{ 2 \sin 46.2915 (t-\tau) - 46.2915 \cos 46.2915 (t-\tau) \right\} \right]_{\tau=0}^{\tau=t}$$

$$= 0.03194 e^{-2t} + 13.7994 (10^{-4}) \sin 46.2915 t - 0.03194 \cos 46.2915 t \text{ radian} \tag{4}$$

4.34



Net compression of spring PQ $= \frac{3 \ell \theta}{4}$ - x(t). Equation of motion for rotation about O:

$$J_0 \ddot{\theta} = -k \frac{\ell \theta}{4} \left(\frac{\ell}{4}\right) - k \left(\frac{3 \ell \theta}{4} - x(t)\right) \left(\frac{3 \ell}{4}\right)$$

$$\text{or} \quad J_0 \ddot{\theta} + \frac{5}{8} k \ell^2 \theta = \frac{3 k \ell}{4} x(t) = \frac{3 k \ell}{4} x_0 e^{-t} \tag{1}$$

For given data, $J_0 = \frac{1}{12} m \ell^2 + m (\frac{\ell}{4})^2 = \frac{7}{48} m \ell^2 = \frac{7}{48} (10)(1^2) = 1.4583$ kg—m^2, $\frac{5}{8} k \ell^2 = \frac{5}{8} (5000)(1^2) = 3125$ N/m, and Eq. (1) becomes:

$$1.4583 \ddot{\theta} + 3125.0 \theta = 37.5 e^{-t} \tag{2}$$

Noting that the system is undamped with

$$\omega_n = \sqrt{\frac{3125.0}{1.4583}} = 46.2915 \text{ rad/sec}$$

the convolution integral, Eq. (4.31), can be used to find the steady state response as:

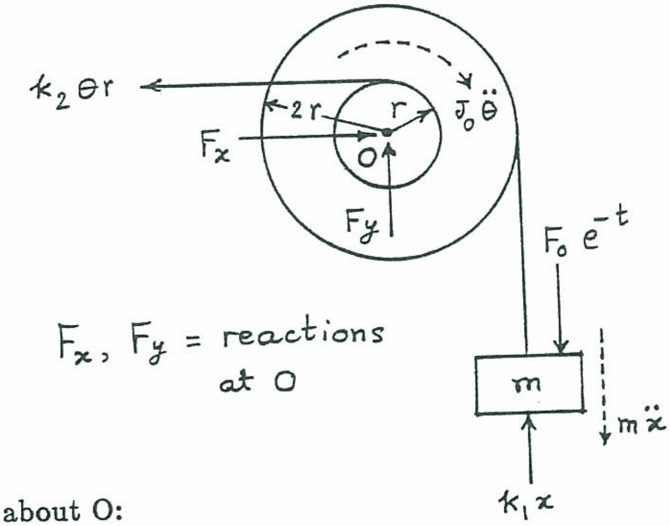
$$\theta(t) = \frac{1}{(1.4583)\,(46.2915)} \int_0^t (37.5\, e^{-\tau}) \sin 46.2915\,(t-\tau)\, d\tau \tag{3}$$

Using Eq. (5) in the solution of Problem 4.32, Eq. (3) can be expressed as

$$\theta(t) = -\,0.5555\, e^{-t} \left[\frac{e^{(t-\tau)}}{1^2 + 46.2915^2} \left\{ \sin 46.2915\,(t-\tau) - 46.2915 \cos\,(t-\tau) \right\} \right]_{\tau=0}^{t}$$

$$= 0.01199\, e^{-t} + 2.591\,(10^{-4}) \sin 46.2915\, t - 0.01199 \cos 46.2915\, t \text{ radian} \tag{4}$$

4.35



Equation of motion for rotation of pulley about O:

$$J_0\, \ddot{\theta} + m\, \ddot{x}\,(2\,r) + k_1\, x\,(2\,r) + k_2\,(\theta\, r)\, r = 2\, r\, F_0\, e^{-t} \tag{1}$$

where $\theta = \dfrac{x}{2\,r}$. Eq. (1) can be rewritten in terms of x only as:

$$\left(\frac{J_0}{2\,r} + 2\, m\, r \right) \ddot{x} + \left(2\, k_1\, r + \frac{1}{2}\, k_2\, r \right) x = 2\, r\, F_0\, e^{-t} \tag{2}$$

For given data, Eq. (2) becomes

$$11.0\, \ddot{x} + 112.5\, x = 5\, e^{-t} \tag{3}$$

Noting that the system is undamped with $\omega_n = \sqrt{\dfrac{112.5}{11.0}} = 3.1980$ rad/sec, the convolution integral, Eq. (4.31), can be used to find the steady state response as:

$$x(t) = -\,0.1421\, e^{-t} \int_0^t e^{(t-\tau)} \sin 3.1980\,(t-\tau))\,(-\,d\tau) \tag{4}$$

Using Eq. (5) in the solution of Problem 4.32, Eq. (4) can be expressed as

$$x(t) = -\frac{0.1421\ e^{-t}}{1^2 + 3.1980^2}\left[e^{(t-\tau)} \left\{ \sin 3.1980\ (t-\tau) - 3.1980 \cos 3.1980\ (t-\tau) \right\} \right]_{\tau=0}^{t}$$

$$= 0.04048\ e^{-t} + 0.01266 \sin 3.1980\ t - 0.04048 \cos 3.1980\ t \text{ m} \qquad (5)$$

4.36 (a) Unit impulse response function for undamped case:

Use $\zeta = 0$ and $\omega_d = \omega_n$ in Eq. (4.25):

$$x(t) = \frac{1}{m\,\omega_n} \sin \omega_n t \qquad (E.1)$$

(b) Unit impulse response function for underdamped case: Eq. (4.25):

$$x(t) = \frac{1}{m\,\omega_d}\, e^{-\zeta \omega_n t} \sin \omega_d t \qquad (E.2)$$

(c) Unit impulse response function for critically damped case:

Free vibration response of a critically damped system is given by Eq. (2.80):

$$x(t) = \{x_0 + (\dot{x}_0 + \omega_n x_0)\, t\}\, e^{-\omega_n t} \qquad (E.3)$$

Using the initial conditions $x_0 = 0$ and $\dot{x}_0 = \frac{1}{m}$, Eq. (E.3) gives

$$x(t) = \frac{t}{m}\, e^{-\omega_n t} \qquad (E.4)$$

(d) Unit impulse response function for an over damped case:

Free vibration response of an overdamped system is given by Eq. (2.81):

$$x(t) = C_1\, e^{(-\zeta + \sqrt{\zeta^2 - 1})\,\omega_n t} + C_2\, e^{(-\zeta - \sqrt{\zeta^2-1})\,\omega_n t} \qquad (E.5)$$

where

$$C_1 = \frac{x_0\, \omega_n (\zeta + \sqrt{\zeta^2 - 1})\ \dot{x}_0}{2\, \omega_n \sqrt{\zeta^2 - 1}}\ ; \quad C_2 = \frac{-x_0\, \omega_n (\zeta - \sqrt{\zeta^2-1}) - \dot{x}_0}{2\, \omega_n \sqrt{\zeta^2 - 1}}$$

For the initial conditions $x_0 = 0$ and $\dot{x}_0 = \frac{1}{m}$,

C_1 and C_2 become

$$C_1 = \frac{1}{2 m \omega_n \sqrt{\zeta^2 - 1}} \quad ; \quad C_2 = -\frac{1}{2 m \omega_n \sqrt{\zeta^2 - 1}}$$

and hence Eq. (E.5) yields

$$x(t) = \frac{1}{2 m \omega_n \sqrt{\zeta^2 - 1}} \left\{ e^{-\left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t} - e^{-\left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t} \right\} \tag{E.6}$$

4.37 $m = 2$ kg, $c = 4$ N-s/m, $k = 32$ N/m, $\underset{\sim}{F} = 4\,\delta(t)$,

$x_0 = 0.01$ m, $\dot{x}_0 = 1$ m/s

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{32}{2}} = 4 \text{ rad/s}, \quad \zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{k m}} = \frac{4}{2\sqrt{32(2)}} = 0.25$$

$$\omega_d = \sqrt{1 - \zeta^2}\, \omega_n = 3.872983 \text{ rad/s}$$

Underdamped system.

Impulse response is given by Eq. (4.26):

$$x(t) = \frac{\underset{\sim}{F}\, e^{-\zeta \omega_n t}}{m\, \omega_d} \sin \omega_d t$$

$$= 0.516398\, e^{-t} \sin 3.872983\, t \quad \text{m}$$

4.38 The stiffness of the cantilever beam (wing) is given by

$$k = \frac{3EI}{l^3} = \frac{3\,(15 \times 10^9)}{10^3} = 45 \times 10^6 \text{ N/m}$$

System can be modeled as a single degree of freedom undamped system:

$$m \ddot{x} + k x = 0$$

where $m = 2500$ kg, $k = 45 \times 10^6$ N/m, and

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{45 \times 10^6}{2.5 \times 10^3}} = 134.1641 \text{ rad/s}$$

Response of mass due to impulse $\underset{\sim}{F}$ is given by Eq. (4.26) with $\zeta = 0$ and $\omega_d = \omega_n$:

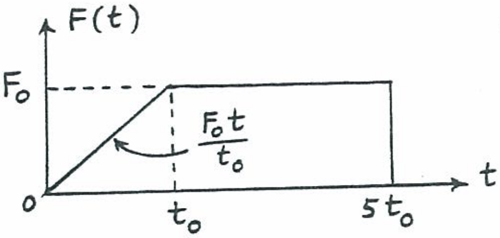
$$x(t) = \frac{\underset{\sim}{F}}{m\,\omega_n} \sin \omega_n t$$

$$= \frac{50}{2500\,(134.1641)} \sin 134.1641\, t$$

$$= 0.000149071 \sin 134.1641\, t \quad m$$

4.39

$$F(t) = \begin{cases} (F_0 t / t_0) & ; \ 0 \le t \le t_0 \\ F_0 & ; \ t_0 \le t \le 5 t_0 \\ 0 & ; \ t > 5 t_0 \end{cases} \quad \text{---}(E_1)$$



Response of the anvil is given by [Eq. (4.31) for an undamped system]:

$$x(t) = \frac{1}{m\omega_n} \int_0^t F(\tau) \sin \omega_n (t-\tau) \, d\tau \qquad (E_2)$$

For $0 \le t \le t_0$:

$$x(t) = \frac{F_0}{m \omega_n t_0} \int_0^t \tau \sin \omega_n (t - \tau) \, d\tau$$

$$= \frac{F_0}{k\, t_0} \left(t - \frac{1}{\omega_n} \sin \omega_n t \right) \qquad (E_3)$$

For $t_0 \le t \le 5 t_0$:

$$x(t) = \frac{1}{m\omega_n} \int_0^t F(\tau) \sin \omega_n (t-\tau) \, d\tau$$

$$= \frac{F_0}{m\omega_n t_0} \int_0^{t_0} \tau \sin \omega_n (t-\tau)\, d\tau + \frac{F_0}{m\omega_n} \int_{t_0}^{t} \sin \omega_n (t-\tau) \, d\tau$$

$$= \frac{F_0}{m\omega_n t_0} \Big[\sin \omega_n t \left\{ \frac{1}{\omega_n^2} \cos \omega_n t_0 + \frac{t_0}{\omega_n} \sin \omega_n t_0 - \frac{1}{\omega_n^2} \right\}$$

$$- \cos \omega_n t \left\{ \frac{1}{\omega_n^2} \sin \omega_n t_0 - \frac{t_0}{\omega_n} \cos \omega_n t_0 \right\} \Big]$$

$$+ \frac{F_0}{m \omega_n} \left\{ \frac{1}{\omega_n} \cos (\omega_n t - \omega_n \tau) \right\}_{t_0}^{t}$$

$$= \frac{F_0}{k\, \omega_n t_0} \left[\sin \omega_n (t - t_0) - \sin \omega_n t + \omega_n t_0 \right] \qquad (E_4)$$

For $t > 5t_0$:

$$x(t) = \frac{1}{m\omega_n}\left[\int_0^{t_0} \frac{F_0 \tau}{t_0} \sin\omega_n(t-\tau)\, d\tau + F_0 \int_{t_0}^{5t_0} \sin\omega_n(t-\tau)\, d\tau\right]$$

$$= \frac{F_0}{m\omega_n^2 t_0}\left[\frac{1}{\omega_n}\sin\omega_n(t-t_0) + t_0\cos\omega_n(t-t_0) - \frac{1}{\omega_n}\sin\omega_n t\right]$$

$$+ \frac{F_0}{m\omega_n}\left[\frac{1}{\omega_n}\cos\omega_n(t-\tau)\right]_{t_0}^{5t_0}$$

$$= \frac{F_0}{k\omega_n t_0}\left[\sin\omega_n(t-t_0) - \sin\omega_n t + t_0\omega_n\cos\omega_n(t-5t_0)\right] \quad (E_5)$$

4.47 $$F(t) = \begin{cases} F_0 & ; \quad 0 \le t \le t_0 \\ 0 & ; \quad t > t_0 \end{cases} \qquad (E_1)$$

Eq. (4.31) gives, for an undamped system,

$$x(t) = \frac{1}{m\,\omega_n} \int_0^t F(\tau)\, \sin \omega_n (t-\tau)\, d\tau \qquad (E_2)$$

For $0 \le t \le t_0$:

$$x(t) = \frac{F_0}{m\,\omega_n} \int_0^t \sin \omega_n (t-\tau)\, d\tau = \frac{F_0}{k}\left(1 - \cos \omega_n t\right) \qquad (E_3)$$

using Eq. (E_5) in the solution of problem 4.27.

For $t > t_0$:

$$x(t) = \frac{F_0}{m\,\omega_n} \int_0^{t_0} \sin \omega_n (t-\tau)\, d\tau$$

Using the relation

$$\int_0^{t_0} \sin \omega_n (t-\tau) \cdot d\tau = \frac{1}{\omega_n}\left\{ \cos \omega_n (t-t_0) - \cos \omega_n t \right\}$$

the solution can be expressed as

$$x(t) = \frac{F_0}{k}\left[\cos \omega_n (t-t_0) - \cos \omega_n t \right] \qquad (E_4)$$

Response spectrum:

For $0 \le t \le t_0$, $$x(t) = \frac{F_0}{k}\left(1 - \cos \omega_n t\right) \qquad (E_5)$$

$$\frac{dx}{dt} = \frac{F_0\,\omega_n}{k} \sin \omega_n t = 0 \Rightarrow \omega_n t_{max} = \pi$$

$$\therefore x_{max} = x(t = t_{max}) = \frac{F_0}{k}\left(1 - \cos \omega_n t_{max}\right) = \frac{2 F_0}{k} \qquad (E_6)$$

For $t > t_0$,

$$x(t) = \frac{F_0}{k}\left\{ \cos \omega_n (t-t_0) - \cos \omega_n t \right\} \qquad (E_7)$$

$$\frac{dx}{dt} = -\frac{F_0\,\omega_n}{k}\left\{ \sin \omega_n (t-t_0) - \sin \omega_n t \right\} = 0$$

$$\Rightarrow \quad \sin \omega_n (t_{max} - t_0) = \sin \omega_n t_{max}$$

i-e, $$\tan \omega_n t_{max} = \left(\frac{\sin \omega_n t_0}{\cos \omega_n t_0 - 1} \right) \qquad (E_8)$$

i.e., $$\sin \omega_n t_{max} = \frac{\sin \omega_n t_0}{\sqrt{2(1 - \cos \omega_n t_0)}} \qquad (E_9)$$

and $$\cos \omega_n t_{max} = \frac{\cos \omega_n t_0 - 1}{\sqrt{2(1 - \cos \omega_n t_0)}} \qquad (E_{10})$$

$$\therefore \quad x_{max} = x(t = t_{max}) = \frac{F_0}{k} \left\{ \cos \omega_n t_{max} \cdot \cos \omega_n t_0 + \sin \omega_n t_{max} \cdot \sin \omega_n t_0 - \cos \omega_n t_{max} \right\}$$

$$= \frac{F_0}{k} \left\{ \frac{(\cos \omega_n t_0 - 1)^2}{\sqrt{2(1 - \cos \omega_n t_0)}} + \frac{\sin^2 \omega_n t_0}{\sqrt{2(1 - \cos \omega_n t_0)}} \right\}$$

$$= \frac{2 F_0}{k} \sin \frac{\omega_n t_0}{2} \qquad (E_{11})$$

Plotting: $\omega_n = \dfrac{2\pi}{\tau_n}$

For $0 \le t \le t_0$, $\quad \omega_n t_{max} = \pi \quad$ or $\quad \dfrac{t_{max}}{\tau_n} = \dfrac{1}{2}$

when $t \le t_0$, $\quad t_{max} = \dfrac{\tau_n}{2} \le t_0$

i-e., $\dfrac{t_0}{\tau_n} \ge \dfrac{1}{2}$

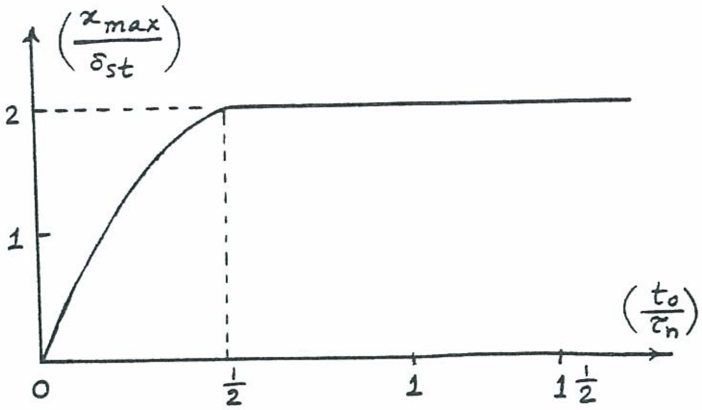
$$\therefore \quad \frac{x_{max}}{\delta_{st}} = 2 \quad \text{for} \quad \frac{t_0}{\tau_n} \ge \frac{1}{2} \qquad (E_{12})$$

For $t > t_0$, $\quad \dfrac{t_0}{\tau_n} < \dfrac{1}{2}$

$$\frac{x_{max}}{\delta_{st}} = 2 \sin \frac{\omega_n t_0}{2} = 2 \sin \frac{2\pi t_0}{2 \tau_n}$$

$$\therefore \quad \frac{x_{max}}{\delta_{st}} = 2 \sin \frac{\pi t_0}{\tau_n} \quad \text{for} \quad \frac{t_0}{\tau_n} < \frac{1}{2} \qquad (E_{13})$$

Eqs. (E_{12}) and (E_{13}) are plotted in the figure.



Response spectrum for a rectangular pulse-type load

4.48 The response is found in problem 4.22.

For $0 \le t \le t_0$:

$$x(t) = \frac{F_0}{k}\left(1 - \cos\omega_n t\right) + \frac{F_0}{2m\omega_n} \cdot \frac{2\omega_n}{\left\{\left(\frac{\pi}{2t_0}\right)^2 - \omega_n^2\right\}} \cdot \left\{\cos\frac{\pi t}{2t_0} - \cos\omega_n t\right\} \qquad (E_1)$$

For x_{max}, $\frac{dx}{dt} = 0$

i.e., $$\frac{F_0\,\omega_n}{k}\sin\omega_n t + \frac{F_0}{m\left\{\left(\frac{\pi}{2t_0}\right)^2 - \omega_n^2\right\}}\left(-\frac{\pi}{2t_0}\sin\frac{\pi t}{2t_0} + \omega_n\sin\omega_n t\right) = 0$$

which can be reduced to the form

$$\sin\omega_n t_{max} = \left[\frac{\pi}{2\omega_n t_0\left\{m\left(\frac{\pi}{2t_0}\right)^2 - m\,\omega_n^2\right\} + k}\right]\sin\frac{\pi\, t_{max}}{2t_0} \qquad (E_2)$$

Once t_{max} is known from (E_2), Eq. (E_1) can be used to find x_{max} as:

$$x_{max} = x(t = t_{max}) = \frac{F_0}{k}\left(1 - \cos\omega_n t_{max}\right) + \frac{F_0}{m\left\{\left(\frac{\pi}{2t_0}\right)^2 - \omega_n^2\right\}}\left(\cos\frac{\pi\, t_{max}}{2t_0} - \cos\omega_n t_{max}\right) \qquad (E_3)$$

For $t > t_0$:

$$x(t) = \frac{F_0}{k} \cos \omega_n (t - t_0) - \sin \omega_n (t - t_0) \left\{ \frac{F_0}{2m\,\omega_n \left(\frac{\pi}{2t_0} - \omega_n \right)} + \frac{F_0}{2m\,\omega_n \left(\frac{\pi}{2t_0} + \omega_n \right)} \right\}$$

$$+ \cos \omega_n t \left\{ - \frac{F_0}{k} - \frac{F_0}{2m\,\omega_n \left(\frac{\pi}{2t_0} - \omega_n \right)} + \frac{F_0}{2m\,\omega_n \left(\frac{\pi}{2t_0} + \omega_n \right)} \right\} \quad (E_4)$$

For t_{max}, $\frac{dx}{dt} = 0$

i.e.,

$$- \frac{F_0\,\omega_n}{k} \sin \omega_n (t_{max} - t_0) - \frac{F_0\,\pi\,\omega_n \cos \omega_n (t_{max} - t_0)}{2m\,\omega_n\,t_0 \left\{ \left(\frac{\pi}{2t_0} \right)^2 - \omega_n^2 \right\}}$$

$$+ \frac{F_0 \left(\frac{\pi}{2t_0} \right)^2 \omega_n}{k \left\{ \left(\frac{\pi}{2t_0} \right)^2 - \omega_n^2 \right\}} \sin \omega_n t_{max} = 0 \quad (E_5)$$

once t_{max} is found by solving Eq. (E_5), x_{max} can be found from Eq. (E_4) as

$$x_{max} = x(t = t_{max}) = \frac{F_0}{k} \cos \omega_n (t_{max} - t_0)$$

$$- \frac{\pi\,F_0}{2\,\omega_n\,m\,t_0 \left\{ \left(\frac{\pi}{2t_0} \right)^2 - \omega_n^2 \right\}} \sin \omega_n (t_{max} - t_0)$$

$$- \frac{F_0 \left(\frac{\pi}{2t_0} \right)^2}{k \left\{ \left(\frac{\pi}{2t_0} \right)^2 - \omega_n^2 \right\}} \cos \omega_n t_{max} \quad (E_6)$$

Eqs. (E_3) and (E_6) can be used to plot x_{max} versus ω_n to get the displacement response spectrum.

4.49 Base acceleration $= \ddot{y}(t) = a_0 \left(1 - \sin \frac{\pi t}{2 t_0} \right) \quad (E_1)$

For an undamped system, the relative displacement is given by Eq. (4.34):

$$z(t) = - \frac{1}{\omega_n} \int_0^t \ddot{y}(\tau) \sin \omega_n (t - \tau)\, d\tau$$

$$= - \frac{1}{\omega_n} \left[\int_0^t a_0 \sin \omega_n (t - \tau)\, d\tau - \int_0^t a_0 \sin \frac{\pi t}{2 t_0} \sin \omega_n (t - \tau)\, d\tau \right] \quad (E_2)$$

Here

$$\int_0^t \sin \omega_n (t-\tau)\, d\tau = \left(\frac{1-\cos \omega_n t}{\omega_n}\right) \qquad (E_3)$$

from Eq. (E_5) in the solution of problem 4.27.

and

$$\int_0^t \sin \frac{\pi \tau}{2t_0} \left(\sin \omega_n t \cos \omega_n \tau - \cos \omega_n t \sin \omega_n \tau\right) d\tau$$

$$= \sin \omega_n t \left\{ -\frac{\cos\left(\frac{\pi}{2t_0} - \omega_n\right)\tau}{2\left(\frac{\pi}{2t_0} - \omega_n\right)} - \frac{\cos\left(\frac{\pi}{2t_0} + \omega_n\right)\tau}{2\left(\frac{\pi}{2t_0} + \omega_n\right)} \right\}_0^t$$

$$- \cos \omega_n t \left\{ \frac{\sin\left(\frac{\pi}{2t_0} - \omega_n\right)\tau}{2\left(\frac{\pi}{2t_0} - \omega_n\right)} - \frac{\sin\left(\frac{\pi}{2t_0} + \omega_n\right)\tau}{2\left(\frac{\pi}{2t_0} + \omega_n\right)} \right\}_0^t \qquad (E_4)$$

Thus the solution, Eq. (E_2), can be finally expressed as

$$z(t) = -\frac{a_0}{\omega_n}\left(1 - \frac{\cos \omega_n t}{\omega_n}\right) - \frac{a_0}{\omega_n}\left\{ \sin \omega_n t \cdot \left[\frac{\cos\left(\frac{\pi}{2t_0} - \omega_n\right)t}{2\left(\frac{\pi}{2t_0} - \omega_n\right)} + \frac{\cos\left(\frac{\pi}{2t_0} + \omega_n\right)t}{2\left(\frac{\pi}{2t_0} + \omega_n\right)} - 2 \right] + \cos \omega_n t \cdot \left[\frac{\sin\left(\frac{\pi}{2t_0} - \omega_n\right)t}{2\left(\frac{\pi}{2t_0} - \omega_n\right)} - \frac{\sin\left(\frac{\pi}{2t_0} + \omega_n\right)t}{2\left(\frac{\pi}{2t_0} + \omega_n\right)} \right] \right\} \qquad (E_5)$$

4.50 During $0 \le t \le t_0$:

$$x(t) = \frac{F_0}{k}\left(1 - \frac{t}{t_0} - \cos \omega_n t + \frac{1}{\omega_n t_0} \sin \omega_n t\right) \qquad (E_1)$$

$\dot{x}(t) = 0$ gives $\omega_n t_0 \sin \omega_n t_m = 1 - \cos \omega_n t_m$

i.e. $\omega_n t_m = 2 \tan^{-1}(\omega_n t_0)$ $\qquad (E_2)$

(E_1) becomes

$$\frac{x_m}{(F_0/k)} = 1 - \frac{t_m}{t_0} - \cos \omega_n t_m + \frac{1}{\omega_n t_0} \sin \omega_n t_m \qquad (E_3)$$

where t_m is given by (E_2).

During $t > t_0$:

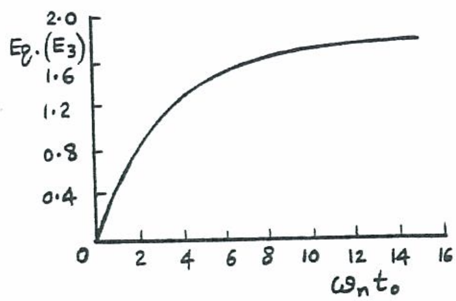
$$x(t) = \frac{F_0}{k\,\omega_n t_0}\left[(1 - \cos \omega_n t_0) \sin \omega_n t - (\omega_n t_0 - \sin \omega_n t_0) \cos \omega_n t\right] \qquad (E_4)$$

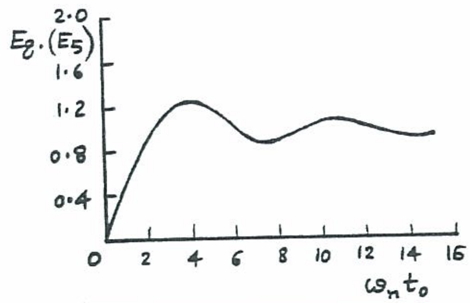
i.e. $\dfrac{x(t)\; k\,\omega_n t_0}{F_0} = \underset{\sim}{x}(t) = A \sin \omega_n t + B \cos \omega_n t$

where $A = 1 - \cos \omega_n t_0$; $B = -(\omega_n t_0 - \sin \omega_n t_0)$

Since $\underset{\sim}{x}\big|_{max} = \sqrt{A^2 + B^2}$,

$$\frac{x_m}{(F_0/k)} = \frac{1}{\omega_n t_0}\left[(1 - \cos \omega_n t_0)^2 + (\omega_n t_0 - \sin \omega_n t_0)^2\right]^{1/2} \qquad (E_5)$$





4.51 From Example 4.13, the response of the building frame is given by

$$x(t) = \frac{F_0}{k}\left[1 - \frac{t}{t_0} - \cos\omega_n t + \frac{1}{\omega_n t_0}\sin\omega_n t\right], \quad 0 \le t \le t_0 \qquad (E_1)$$

and

$$x(t) = \frac{F_0}{k\,\omega_n t_0}\left[(1-\cos\omega_n t_0)\sin\omega_n t - (\omega_n t_0 - \sin\omega_n t_0)\cos\omega_n t\right], \quad t > t_0 \qquad (E_2)$$

(i) For $0 \le t \le t_0$:

For $x(t)$ to be maximum, the quantity inside square brackets in (E_1) must be maximum. This implies that

$$\frac{d}{dt}\left[1 - \frac{t}{t_0} - \cos\omega_n t + \frac{1}{\omega_n t_0}\sin\omega_n t\right] = 0$$

i.e., $\quad \omega_n t_0 \sin\omega_n t = 1 - \cos\omega_n t$

i.e., $\quad \omega_n t_0 \cos\frac{\omega_n t}{2} = \sin\frac{\omega_n t}{2}$

i.e., $\quad \tan\frac{\omega_n t}{2} = \omega_n t_0 \qquad (E_3)$

Thus, if $x(t)$ attains its maximum value at $t = t_{max}$, Eq. (E_3) gives

$$t_{max} = \frac{2}{\omega_n}\tan^{-1}(\omega_n t_0) \qquad (E_4)$$

Once t_{max} is known from (E_4), (E_1) gives

$$x_{max} = x(t = t_{max}) = \frac{F_0}{k}\left\{1 - \frac{t_{max}}{t_0} - \cos\omega_n t_{max} + \frac{1}{\omega_n t_0}\sin\omega_n t_{max}\right\} \qquad (E_5)$$

(ii) For $t > t_0$:

For $x(t)$ to be maximum, the quantity inside square brackets in Eq. (E_2) must be maximum. This implies that

$$\frac{d}{dt}\left[(1-\cos\omega_n t_0)\sin\omega_n t - (\omega_n t_0 - \sin\omega_n t_0)\cos\omega_n t\right] = 0$$

i.e.,

$$(1-\cos\omega_n t_0)\,\omega_n \cos\omega_n t + (\omega_n t_0 - \sin\omega_n t_0)\,\omega_n \sin\omega_n t = 0$$

i.e.,

$$\tan\omega_n t = -\left(\frac{1-\cos\omega_n t_0}{\omega_n t_0 - \sin\omega_n t_0}\right) \qquad (E_6)$$

If $x(t)$ attains its maximum at $t = t_{max}$, Eq. (E_6) gives

$$t_{max} = \frac{1}{\omega_n}\tan^{-1}\left(\frac{-1+\cos\omega_n t_0}{\omega_n t_0 - \sin\omega_n t_0}\right) \qquad (E_7)$$

Once t_{max} is computed from (E_7), (E_2) gives

$$x_{max} = x(t = t_{max}) = \frac{F_0}{k\,\omega_n t_0}\left[(1-\cos\omega_n t_0)^2 + (\omega_n t_0 - \sin\omega_n t_0)^2\right]^{\frac{1}{2}} \qquad (E_8)$$

Given data:

$m = 5000$ kg, $F_0 = 4 \times 10^6$ N, $t_0 = 0.4$ sec, $x_{max} \le 0.01$ m.

$\omega_n = \sqrt{k/m} = 0.01414\sqrt{k}$

Procedure:

1. Assume a series of values of k.
2. Find t_{max} using Eqs. (E_4) and (E_7).
3. Find x_{max} using Eqs. (E_5) and (E_8).
4. Select k such that $x_{max} \le 0.01$ m in Eq. (E_5) or (E_8).

Sample computer program and results are shown below.

```
      XM=5000.0
      FO=4.0E+6
      TO=0.4
      XK=0.0
      DO 10 I=1,100
      XK=XK+1.0E+7
      OMN=0.01414*SQRT(XK)
      TMAX1=(2.0/OMN)*ATAN(OMN*TO)
      XMAX1=(FO/XK)*(1.0-(TMAX1/TO)-COS(OMN*TMAX1)+SIN(OMN*TMAX1)/
     2 (OMN*TO))
      XNR=-(1.0-COS(OMN*TO))
      XDR=(OMN*TO-SIN(OMN*TO))
      TMAX2=ATAN(XNR/XDR)/OMN
      X1=(1.0-COS(OMN*TO))**2
      X2=(OMN*TO-SIN(OMN*TO))**2
      X3=(X1+X2)**0.5
      XMAX2=X3*FO/(XK*OMN*TO)
      PRINT 5,I,XK,TMAX1,XMAX1,TMAX2,XMAX2
5     FORMAT(I5,2X,E15.4,2X,2E12.4,2X,2E12.4)
10    CONTINUE
      STOP
      END
```

1	0. 1000E+08	0. 6776E-01	0. 7322E+00	-0. 5134E-03	0. 41[illegible]E+00
2	0. 2000E+08	0. 4843E-01	0. 3758E+00	-0. 8204E-05	0. 19[illegible]E+00
3	0. 3000E+08	0. 3973E-01	0. 2534E+00	-0. 3859E-04	0. 13[illegible]E+00
4	0. 4000E+08	0. 3450E-01	0. 1914E+00	-0. 4108E-03	0. 10[illegible]E+00
5	0. 5000E+08	0. 3092E-01	0. 1538E+00	-0. 4234E-03	0. 78[illegible]E-01
⋮					

These results indicate that the required stiffness is $k = 8 \times 10^8$ N/m.

4.52 Let d = thickness of bracket. Then, from Example 4.18; self weight of beam = w = 0.5 d lb, total weight at free end of beam = W = 0.5 d + 0.4 lb, moment of inertia of beam cross section = I = 0.04167 d^3 in^4, static deflection of beam under W:

$$\delta_{st} = \frac{W\,\ell^3}{3\,E\,I} = \frac{(0.5\,d + 0.4)}{d^3}\,7.9994\,(10^{-4})\ \text{in}$$

We need to use a trial and error procedure to find the correct value of d.

Let d = 1 in:

$w = 0.5$ lb, $W = 0.9$ lb, $I = 0.04167\ in^4$, $\delta_{st} = 0.9\,(7.9994)\,(10^{-4}) = 7.19946\,(10^{-4})$ in,

$$\tau_n = 2\,\pi\sqrt{\frac{\delta_{st}}{g}} = \sqrt{\frac{7.19946\,(10^{-4})}{386.4}} = 0.008577\ \text{sec}$$

$$\frac{t_0}{\tau_n} = \frac{0.1}{0.008577} = 11.6591$$

From Fig. 4.65, shock amplification factor (A_a) corresponding to $t_0/\tau_n = 11.6591$ is $A_a \approx 2.0$.

Dynamic load at end of cantilever = $P_d = A_a\,M\,a_s = (2.0)\,(0.9/g)\,(100g) = 180.0$ lb.

$$\sigma_{max} = \frac{M_b\,c}{I} = \frac{(180\,(10))\,(1.0/2)}{0.04167} = 21598.2721\ \text{lb/in}^2$$

Since this is smaller than the permissible value of 26000 psi, we choose a smaller value of d next.

Let d = 0.9 in:

$$w = 0.45\ \text{lb},\ W = 0.85\ \text{lb},\ I = 0.03038\ \text{in}^4,\ \delta_{st} = \frac{0.85\,(7.9994\,(10^{-4}))}{0.9^3} = 9.3271\,(10^{-4})\ \text{in}$$

$$\tau_n = 2\,\pi\sqrt{\frac{\delta_{st}}{g}} = 2\,\pi\sqrt{\frac{9.3271\,(10^{-4})}{386.4}} = 0.009762\ \text{sec}$$

$$\frac{t_0}{\tau_n} = \frac{0.1}{0.009762} = 10.2438$$

$A_a \approx 2.0$, $P_d = A_a\,M\,a_s = (2.0)\,(0.85/g)\,(100g) = 170.0$ lb

$$\sigma_{max} = \frac{M_b\,c}{I} = \frac{(170\,(10))\,(0.9/2)}{0.03038} = 25181.0402\ \text{lb/in}^2$$

Since this stress is close to the maximum permissible value, we take d = 0.9 in.

4.53 Let d = thickness of bracket. Then from Example 4.18, self weight of beam = w = 0.5 d lb, total weight at free end of beam = W = 0.5 d + 0.4 lb, moment of inertia of beam cross section = I = 0.04167 d^3 in^4, static deflection of beam under W:

$$\delta_{st} = \left(\frac{0.5\ d + 0.4}{d^3} \right) 7.9994\ (10^{-4})\ \text{in}$$

We need to use a trial and error procedure to find the correct value of d. However, since the shock amplification factor, for large values of t_0/τ_n, for the triangular pulse of Fig. 4.66 is similar to that of the pulse shown in Fig. 4.15, we start with d = 0.6 in. This gives:
w = 0.3 lb, W = 0.7 lb, I = (0.04167) (0.216) = 0.009001 in^4, $\delta_{st} = \frac{0.7}{0.6^3}\ (7.9994\ (10^{-4})) = 25.9240\ (10^{-4})$ in, $\tau_n = 2\ \pi \sqrt{\frac{25.9240\ (10^{-4})}{386.4}} =$ 0.01627 sec, t_0/τ_n = (0.1/0.01627) = 6.1445. From Fig. 4.66, we find shock amplification factor as $A_a \approx 1.1$, dynamic load at end of beam = $P_d = A_a\ M\ a_a$ = (1.1) (0.7/g) (100g) = 77.0 lb, maximum bending stress at root of beam = $\sigma_{max} = \frac{M_b\ c}{I}$ = (77(10)) (0.6/2) / (0.009001) = 25663.8151 psi. Since this stress is very close to the maximum permissible value, we select d = 0.6 in as the design.

4.54 Let d = thickness of bracket (beam). Self weight of beam = w = d (1/2) (16) (0.1) = 0.8 d lb, total load at middle of beam = W = (1 + 0.8 d) lb, area moment of inertia of beam cross section = I = $\frac{1}{12}\ (\frac{1}{2})\ d^3\ in^4$. Static deflection of beam at middle due to W:

$$\delta_{st} = \frac{W\ \ell^3}{192\ E\ I} = \frac{(1 + 0.8\ d)\ (16^3)}{192\ (10^7)\ (0.04167\ d^3)} = (1 + 0.8\ d)\ (51.1959\ (10^{-6}))\ \text{in}$$

We need to use a trial and error procedure to determine the correct value of d.

Let d = 0.4 in:

w = 0.32 lb, W = 1.32 lb, I = 0.002667 in^4, δ_{st} = 67.5786 (10^{-6}) in,

$$\tau_n = 2\ \pi \sqrt{\frac{67.5786\ (10^{-6})}{386.4}} = 0.002628\ \text{sec}$$

$$\frac{t_0}{\tau_n} = \frac{0.1}{0.002628} = 38.0569$$

From Fig. 4.15(b), $A_a \approx 1.1$, and the dynamic load on beam is given by $P_d = A_a\ M\ a_s$ where M = total mass of beam and a_s = acceleration due to shock = 100 g. Thus P_d = (1.1) (1.32/g) (100 g) = 145.2 lb. Maximum bending moment in a fixed-fixed beam due to load (F) at the middle is given by $M_b = \frac{F\ \ell}{8}$ so that

$$\sigma_{max} = \frac{M_b\ c}{I} = \frac{\left(\frac{145.2\ (16)}{8} \right) (\frac{0.4}{2})}{0.002667} = 21777.2778\ \text{lb/in}^2$$

Since this stress is smaller than the maximum permissible value of 26000 psi, we next select a smaller value of d.

Let d = 0.35 in:

$w = 0.28$ lb, $W = 1.28$ lb, $I = 0.001787$ in^4, $\delta_{st} = 65.5307\ (10^{-6})$ in,

$$\tau_n = 2\,\pi \sqrt{\frac{65.5307\ (10^{-6})}{386.4}} = 0.002587 \text{ sec}$$

$$\frac{t_0}{\tau_n} = \frac{0.1}{0.002587} = 38.6469$$

From Fig. 4.15(b), $A_a \approx 1.1$, $P_d = (1.1)\ (1.28/g)\ (100g) = 140.8$ lb

$$\sigma_{max} = \frac{M_b\ c}{I} = \frac{\left(\frac{140.8\ (16)}{8}\right)\left(\frac{0.35}{2}\right)}{0.001787} = 27576.9446 \text{ lb/in}^2$$

Since this stress exceeds the permissible value, we increase the value of d.

Let d = 0.37 in:

$w = 0.296$ lb, $W = 1.296$ lb, $I = 0.002111$ in^4, $\delta_{st} = 66.3499\ (10^{-6})$ in

$$\tau_n = 2\,\pi \sqrt{\frac{66.3499\ (10^{-6})}{386.4}} = 0.002604 \text{ sec}$$

$$\frac{t_0}{\tau_n} = 38.4024$$

From Fig. 4.15(b), $A_a \approx 1.1$, $P_d = (1.1)\ (1.296/g)\ (100\ g) = 142.56$ lb, and

$$\sigma_{max} = \frac{M_b\ c}{I} = \frac{\left(\frac{142.56\ (16)}{8}\right)\left(\frac{0.37}{2}\right)}{0.002111} = 24986.8309 \text{ lb/in}^2$$

Since this stress is close to the maximum permissible value, we select the design as d = 0.37 in.

4.55 $W = m\ g = 100000$ lb, $\varsigma = 0.05$, $\sigma_y = 30000$ psi,

σ_{max} = maximum permissible stress $= \frac{\sigma_y}{2} = \frac{30000}{2} = 15000$ psi,

$$\tau_n = \frac{2\,\pi}{\omega_n} = 2\,\pi\sqrt{\frac{m}{k}} = 2\,\pi\sqrt{\frac{100000}{386.4\ k}} = \frac{101.0793}{\sqrt{k}} \text{ sec}$$

We need to use a trial and error procedure to find k.

Let k = 10000 lb/in:

$$k = 10^4 = \frac{3\,E\,I}{\ell^3} = \frac{3\,(30\,(10^6))\,I}{600^3}$$

$$I = 24000 \text{ in}^4 = \frac{\pi}{64}\, d_o^4\, (0.5904) \text{ with } \frac{d_i}{d_o} = 0.8$$

$$d_o^4 = 82.8121\,(10^4) \text{ in}^4$$

$$d_o = 30.1664 \text{ in} \; ; \; d_i = 24.1331 \text{ in}$$

$$\tau_n = \frac{101.0793}{\sqrt{10^4}} \approx 1 \text{ sec}$$

From Fig. 4.18, for $\tau_n = 1$ sec and $\varsigma = 0.05$, we find $S_v = 25$ in/sec, $S_d = 4.2$ in and $S_a = 0.42$ g.

Maximum shear force in column:

$$F_{max} = \frac{W}{g}\, S_a = \frac{100000}{g}\,(0.42g) = 42000 \text{ lb}$$

Maximum bending moment:

$$M_b = F_{max}\, h = (42000)\,(50\,(12)) = 25.2\,(10^6) \text{ lb—in}$$

Maximum bending stress:

$$\sigma_b = \frac{M_b\, c}{I} = \frac{(25.2\,(10^6))\,(30.1664/2)}{24\,(10^3)} = 15837.36 \text{ lb/in}^2$$

Since this stress is slightly smaller than the maximum permissible value, we choose a larger value of k.

Let k = 20000 lb/in:

$$k = 2\,(10^4) = \frac{3\,(30\,(10^6))\,I}{600^3}$$

$$I = 48000 \text{ in}^4 = \frac{\pi}{64}\, d_o^4\, (0.5904)$$

$$d_o^4 = 165.6243\,(10^4) \text{ in}^4$$

$$d_o = 35.8741 \text{ in} \; ; \; d_i = 28.6993 \text{ in}$$

$$\tau_n = \frac{101.0793}{\sqrt{20000}} = 0.7147 \text{ sec}$$

From Fig. 4.18, we find

$$S_v = 26 \text{ in/sec}, \; S_d = 3 \text{ in}, \; S_a = 0.6 \text{ g}$$

Maximum shear force in column:

$$F_{max} = \frac{W}{g}\, S_a = \frac{10^5}{g}\,(0.6g) = 60000 \text{ lb}$$

Maximum bending moment:

$$M_b = F_{max}\, h = (60000)\,(600) = 36\,(10^6) \text{ lb—in}$$

Maximum bending stress:

$$\sigma_b = \frac{M_b\, c}{I} = \frac{(36\,(10^6))\,(35.8741/2)}{48\,(10^3)} = 13452.7875 \text{ lb/in}^2$$

Since this stress is less than the maximum permissible value, we choose the inner and outer diameters of the column as $d_i = 28.6993$ in and $d_o = 35.8741$ in.

4.56 m g = 5000 lb, $\varsigma = 0.02$. From Fig. 4.19, in order to have $S_a \approx 1$ g, we need to have $\tau_n = 0.2$ sec. Thus

$$\tau_n = 0.2 = \frac{2\pi}{\omega_n} \quad ; \quad \omega_n = 31.416 \text{ rad/sec} = \sqrt{\frac{k}{m}}$$

$$k = (31.416)^2 \text{ m} = (31.416^2)\,(5000/386.4) = 12771.2870 \text{ lb/in}$$

4.58 Equation of motion is $\ddot{x} + \omega_n^2 x = \frac{F_0}{m} e^{i\omega t}$

For zero initial conditions

$$(s^2 + \omega_n^2)\,\bar{x}(s) = \frac{F_0}{m} \cdot \frac{1}{s - i\omega} \quad ; \quad \bar{x}(s) = \frac{F_0}{m} \cdot \frac{1}{(s^2+\omega_n^2)} \cdot \frac{s + i\omega}{(s^2+\omega^2)}$$

Inverse Laplace transformation gives

$$x(t) = \frac{F_0}{m} \frac{1}{(\omega_n^2 - \omega^2)} \left\{ e^{i\omega t} - \left(\cos\omega_n t + \frac{i\omega}{\omega_n} \sin\omega_n t \right) \right\}$$

The terms containing $\cos\omega_n t$ and $\sin\omega_n t$ denote the transient part of the response and hence can be neglected. Thus the steady state response can be expressed as

$$x(t) = \frac{F_0}{k} \left(\frac{1}{1 - r^2} \right) e^{i\omega t} \qquad \text{where } r = \omega/\omega_n .$$

4.59 $\bar{F}(s) = \frac{F_0}{s}$

The Laplace transform of the response can be written as

$$\bar{x}(s) = \frac{F_0}{ms\,(s^2 + 2\zeta\omega_n s + \omega_n^2)} + \left(\frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \right) x_0$$

$$+ \left(\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} \right) \dot{x}_0$$

Inverse transformation gives

$$x(t) = \frac{F_0}{m} \left\{ 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \phi_1) \right\} + \frac{x_0}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi_1)$$

$$+ \frac{\dot{x}_0}{\omega_d} e^{-\zeta\omega_n t} \sin\omega_d t$$

where $\phi_1 = \cos^{-1}(\zeta)$.

4.60 The forcing function can be expressed as

$$F(t) = F_0 \{ u(t) - u(t - t_0) \}$$

where $u(t-\tau)$ is the unit step function applied at $t = \tau$. The Laplace transform of $F(t)$ is

$$\bar{F}(s) = F_0 \left(\frac{1}{s} - \frac{1}{s} e^{-st_0} \right)$$

The equation of motion $m\ddot{x} + kx = F(t)$ gives

$$\bar{x}(s) = \frac{F_0}{s}\left(1 - e^{-st_0}\right)\frac{1}{ms^2 + k} = \frac{F_0}{m}\left\{\frac{1}{s(s^2+\omega_n^2)} - \frac{e^{-st_0}}{s(s^2+\omega_n^2)}\right\}$$

Since $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\} = \frac{1}{a^2}\left(1 - \cos at\right)$,

$$x(t) = \frac{F_0}{m\omega_n^2}\left(1 - \cos\omega_n t\right)\cdot u(t) - \frac{F_0}{m\omega_n^2}\left\{1 - \cos\omega_n(t - t_0)\right\} u(t - t_0)$$

Hence

$$x(t) = \begin{cases} \dfrac{F_0}{m\omega_n^2}\left(1 - \cos\omega_n t\right) & \text{for } 0 \le t \le t_0 \\[2ex] \dfrac{F_0}{m\omega_n^2}\left\{\cos\omega_n(t - t_0) - \cos\omega_n t\right\} & \text{for } t \ge t_0 \end{cases}$$

4.84

Method 1:

$$x_j = \frac{1}{k} \sum_{i=1}^{j-1} \Delta F_i \left\{ 1 - \cos \omega_n (t_j - t_i) \right\}$$

$$\dot{x}_j = \frac{1}{k} \sum_{i=1}^{j-1} \Delta F_i \, \omega_n \sin \omega_n (t_j - t_i)$$

Method 2:

$$x_j = \frac{F_j}{k} \left[1 - \cos \omega_n \Delta t_j \right] + x_{j-1} \cos \omega_n \Delta t_j + \frac{\dot{x}_{j-1}}{\omega_n} \sin \omega_n \Delta t_j$$

$$\dot{x}_j = \frac{F_j \omega_n}{k} \sin \omega_n \Delta t_j + \omega_n \left\{ - x_{j-1} \sin \omega_n \Delta t_j + \frac{\dot{x}_{j-1}}{\omega_n} \cos \omega_n \Delta t_j \right\}$$

Method 3:

$$x_j = \frac{\Delta F_j}{k \, \Delta t_j} \left\{ \Delta t_j - \frac{1}{\omega_n} \sin \omega_n \Delta t_j \right\} - \frac{F_{j-1}}{k} \left(1 - \cos \omega_n \Delta t_j \right) + x_{j-1} \cos \omega_n \Delta t_j$$

$$+ \frac{\dot{x}_{j-1}}{\omega_n} \sin \omega_n \Delta t_j$$

$$\dot{x}_j = \frac{\Delta F_j}{k\,\Delta t_j}\left(1 - \cos\omega_n \Delta t_j\right) + \frac{F_{j-1}}{k}\,\omega_n \sin\omega_n \Delta t_j + \dot{x}_{j-1}\cos\omega_n \Delta t_j - \omega_n x_{j-1}\sin\omega_n \Delta t_j$$

VALUE OF	Method 1 Step variation with larger value	Method 1 Step variation with smaller value	Method 2 Step variation with mid-value	Method 3 Linear variation
I	X(I)	X(I)	X(I)	X(I)
2	0.489135E-01	0.412870E-01	0.451034E-01	0.463829E-01
3	0.183327E+00	0.153639E+00	0.168413E+00	0.170863E+00
4	0.374871E+00	0.311494E+00	0.342969E+00	0.346380E+00
5	0.590262E+00	0.485757E+00	0.537537E+00	0.541621E+00
6	0.794774E+00	0.646981E+00	0.720014E+00	0.724433E+00
7	0.955998E+00	0.768556E+00	0.860896E+00	0.865287E+00
8	0.104732E+01	0.829582E+00	0.936446E+00	0.940461E+00
9	0.105081E+01	0.817133E+00	0.931268E+00	0.934605E+00
10	0.959172E+00	0.727695E+00	0.840009E+00	0.842438E+00
11	0.776638E+00	0.566437E+00	0.668027E+00	0.669410E+00
I	XD(I)	XD(I)	XD(I)	XD(I)
2	0.309017E+00	0.260676E+00	0.284772E+00	0.284646E+00
3	0.539444E+00	0.448685E+00	0.493775E+00	0.493286E+00
4	0.669917E+00	0.547974E+00	0.608327E+00	0.607283E+00
5	0.690013E+00	0.552279E+00	0.620126E+00	0.618406E+00
6	0.601222E+00	0.465650E+00	0.531993E+00	0.529561E+00
7	0.416706E+00	0.301948E+00	0.357498E+00	0.354412E+00
8	0.159909E+00	0.833536E-01	0.119506E+00	0.115921E+00
9	-0.137878E+00	-0.161957E+00	-0.152198E+00	-0.156045E+00
10	-0.440725E+00	-0.402734E+00	-0.423989E+00	-0.427799E+00
11	-0.711751E+00	-0.615408E+00	-0.661862E+00	-0.665302E+00

4.88 $m = 10$ kg, $k = 4000$ N/m, $c = 40$ N-s/m, $\underset{\sim}{F} = 100$ N-s

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4000}{10}} = 20 \text{ rad/s}$$

$$\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{k\,m}} = \frac{40}{2\sqrt{4000\,(10)}} = 0.1$$

$$\omega_d = \sqrt{1-\zeta^2}\;\omega_n = \sqrt{1-0.1^2}\,(20) = 19.899749 \text{ rad/s}$$

Assuming that impact is given at $t=0$, we find the response of the system as (from Eq. (4.26)):

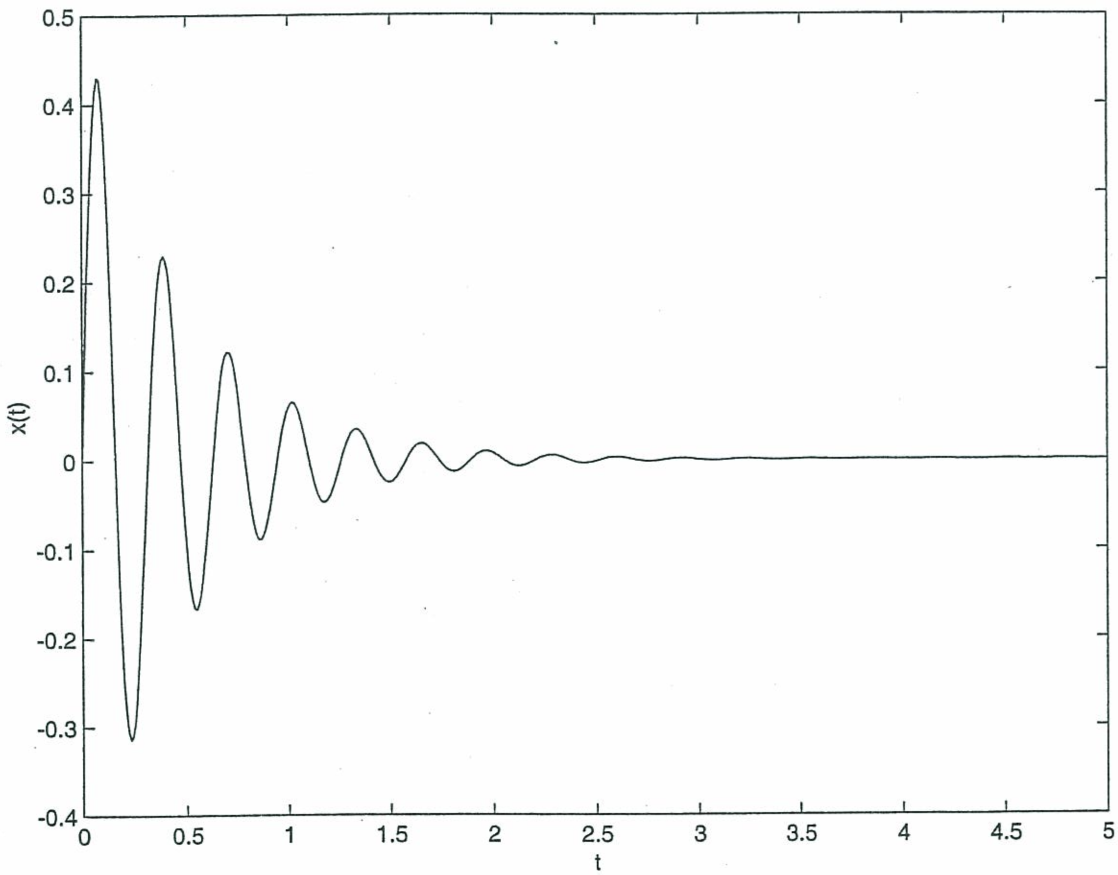
$$x_1(t) = \underset{\sim}{F}\,\frac{e^{-\zeta\omega_n t}}{m\,\omega_d}\,\sin\omega_d t$$

$$= (100)\,\frac{e^{-(0.1)(20)\,t}}{10\,(19.899749)}\,\sin 19.899749\,t$$

$$\therefore \quad x_1(t) = 0.502519 \; e^{-2t} \; \sin 19.899749\, t \qquad (E.1)$$

Plotting of Eq. (E.1) using MATLAB:

```
% Ex4_88.m
for i = 1: 501
    t(i) = 5*(i-1)/500;
    x(i) = 0.502519 * exp(-2.0*t(i)) * sin( 19.899749*t(i) );
end
plot(t,x);
xlabel('t');
ylabel('x(t)');
```



4.89 $\omega_n = 20$ rad/s, $\zeta = 0.1$, $\omega_d = 19.899749$ rad/s

Response due to $\underset{\sim}{F}_1\, \delta(t)$ is given by Eq. (E.1) of Problem 4.88.

Response due to $\underset{\sim}{F}_2\, \delta(t-0.5)$ can be found from Eqs. (4.27) and (4.26):

$$x_2(t) = \underset{\sim}{F}_2 \; \frac{e^{-\zeta \omega_n (t-\tau)}}{m\, \omega_d} \sin \omega_d (t-\tau) \qquad (E.2)$$

For $\tau = 0.5$, Eq. (E.2) gives

$$x_2(t) = \frac{50 \; e^{-0.1(20)(t-0.5)}}{10\,(19.899749)} \sin 19.899749\,(t-0.5)$$

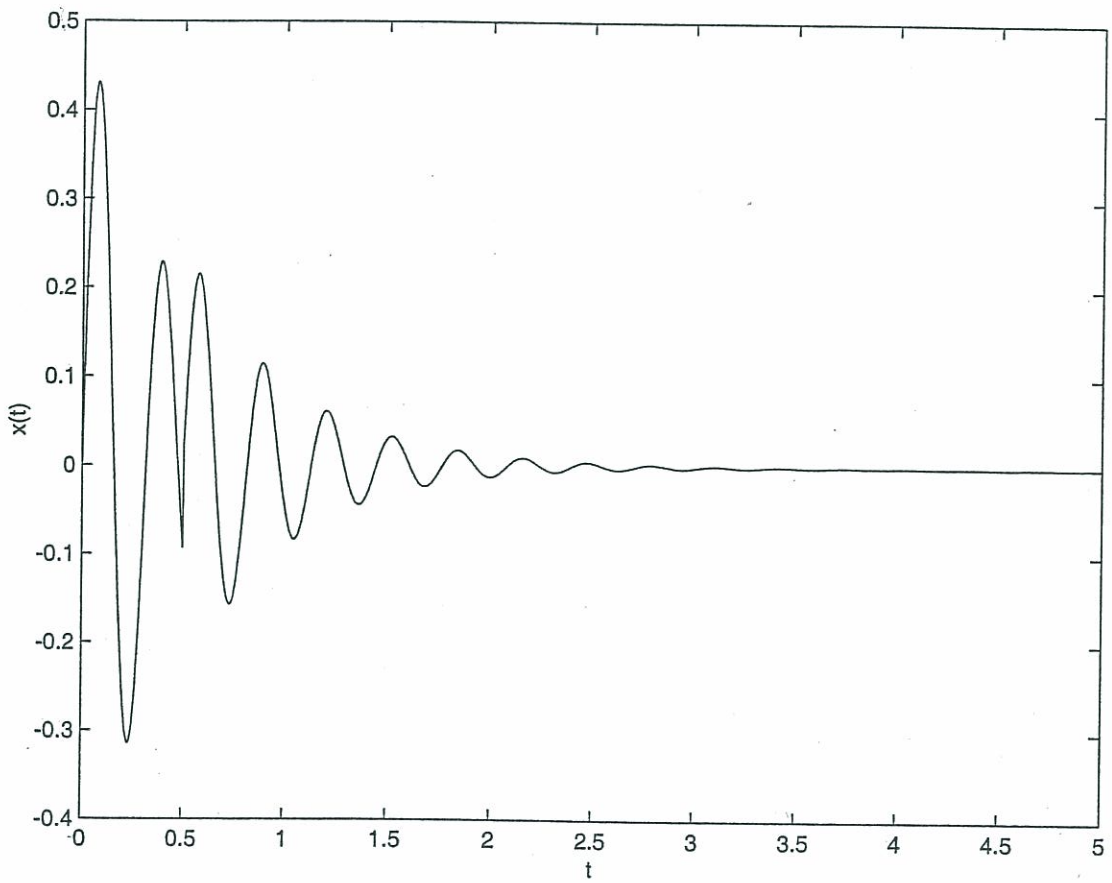
$$= 0.251259 \; e^{-2(t-0.5)} \sin 19.899749\,(t-0.5)$$

The response due to two impacts (in meters) is given by

$$x(t) = \begin{cases} 0.502519 \; e^{-2t} \sin 19.899749\,t \; ; & 0 \le t \le 0.5 \\ 0.251259 \; e^{-2(t-0.5)} \sin 19.899749\,(t-0.5) \; ; & t > 0.5 \end{cases} \quad (E.3)$$

Plotting of Eq. (E.3) using MATLAB:

```
% Ex4_89.m
for i = 1: 1001
    t(i) = (i-1)*5/1000;
    if t(i) <= 0.5
       x(i) = 0.502519 * exp(-2*t(i)) * sin(19.899749*t(i));
    else
       x(i) = 0.251259 * exp(-2*(t(i)-0.5)) * sin(19.899749*(t(i)-0.5));
    end
end
plot(t,x);
xlabel('t');
ylabel('x(t)');
```



4.90 Response due to step load (see Eq. (E.1) of Example 4.11):

$$x(t) = \frac{F_0 \, e^{-\zeta \omega_n t}}{k \sqrt{1-\zeta^2}} \Big\{ -\cos(\omega_d t - \phi) + e^{\zeta \omega_n t_0} \cos[\omega_d (t - t_0) - \phi] \Big\}$$

with $\phi = \tan^{-1} \left\{ \frac{\zeta}{\sqrt{1-\zeta^2}} \right\}$

Data: $m = 100$ kg, $k = 1200$ N/m, $c = 50$ N-s/m, $F_0 = 100$ N, $t_0 = 0.1$ s and 1.5 s

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{12} = 3.464102 \text{ rad/s},$$

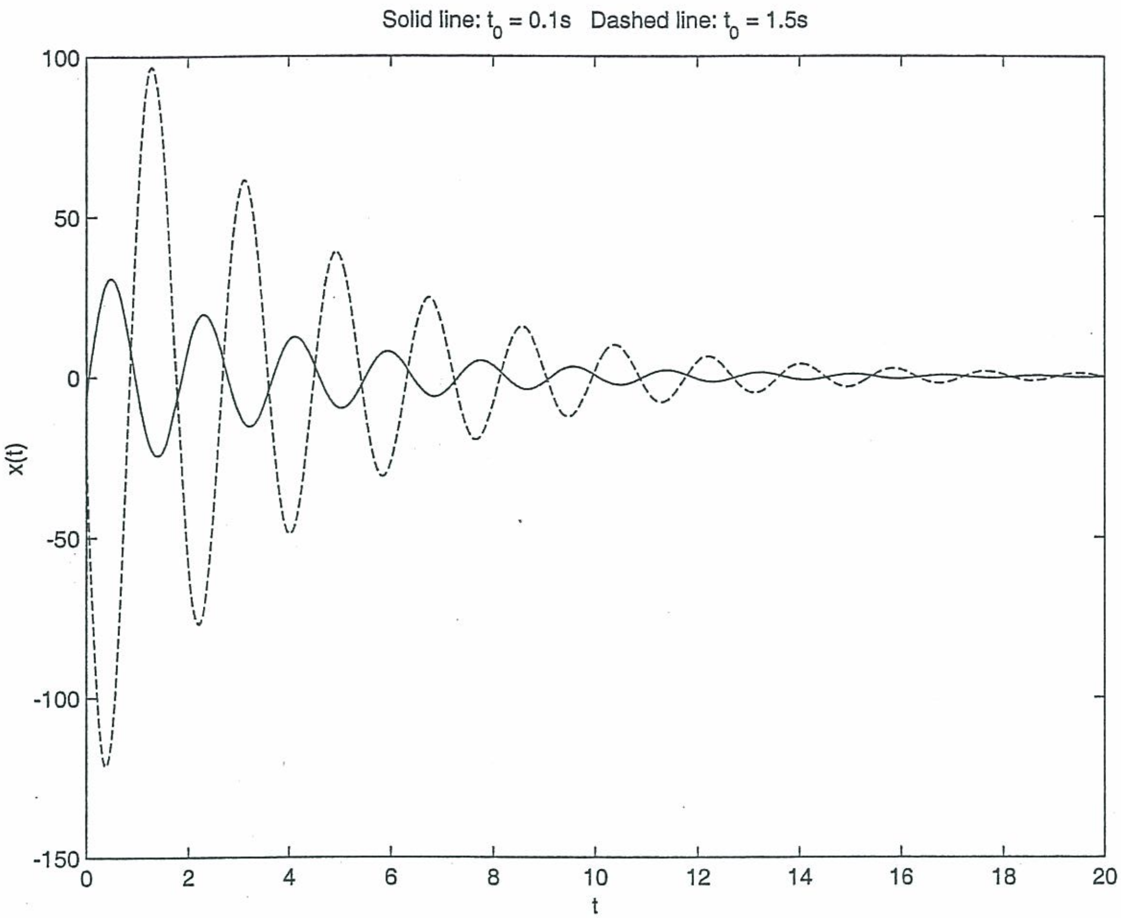
$$\zeta = \frac{c}{2\sqrt{k m}} = \frac{50}{2\sqrt{(1200)(100)}} = \frac{25}{346.410161} = 0.072169$$

$$\omega_d = \sqrt{1-\zeta^2}\, \omega_n = 3.455069 \text{ rad/s}$$

$$\phi = \tan^{-1}\left(\frac{0.072169}{0.997392}\right) = 0.072232 \text{ rad}$$

Plotting:

```
% Ex4_90.m
F0 = 100;
m = 100;
k = 1200;
c = 50;
wn = 3.464102;
zeta = 0.072169;
wd = 3.455069;
phi = 0.072232;
t0 = 0.1;
for i = 1: 501
    t(i) = 20*(i-1)/500;
    x1(i) = F0 * exp(-zeta*wn*t(i)) * ( -cos(wd*t(i)-phi) + ...
        exp(zeta*wn*t0) * cos(wd*(t(i)-t0)-phi) );
end
t0 = 1.5;
for i = 1: 501
    t2(i) = 20*(i-1)/500;
    x2(i) = F0 * exp(-zeta*wn*t2(i)) * ( -cos(wd*t2(i)-phi) + ...
        exp(zeta*wn*t0) * cos(wd*(t2(i)-t0)-phi) );
end
plot(t,x1);
xlabel('t');
ylabel('x(t)');
title('Solid line: t_0 = 0.1s   Dashed line: t_0 = 1.5s');
hold on;
plot(t2,x2,'--');
```



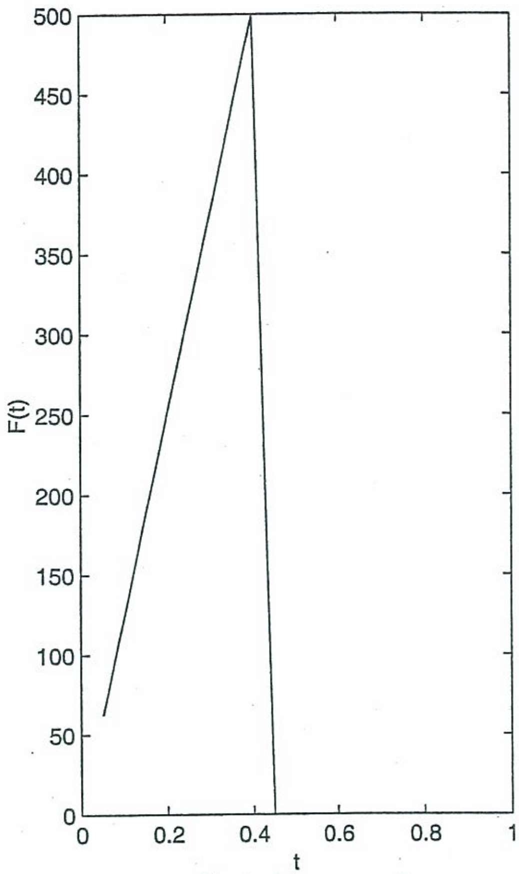
4.91

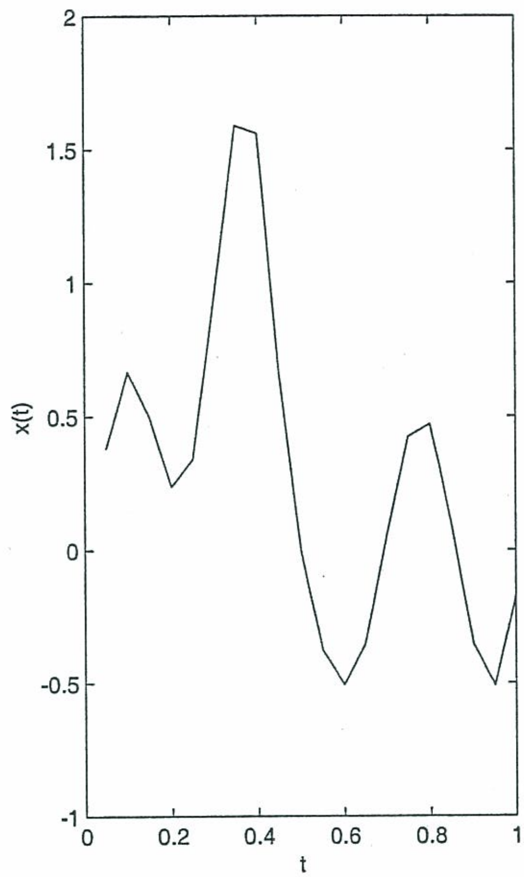
```
%===============================================================
%
%Program4.m
%Main program which calls PERIOD
%
%===============================================================
%Run "Program4" in MATLAB command window. Program4.m and period.m
%should be in the same folder,and set the path to this folder
%following seven lines contain problem-dependent data
xm=1.0;
xk=400.0;
xai=0.125;
n=20;
m=10;
time=1;
f=[62.5 125 187.5 250 312.5 375 437.5 500 zeros(1,12)]';
t=0.05:0.05:1.00;
%end of problem-dependent data
[xsin,xcos,psi,phi,fzero,fc,x,xpc,xps]=period(xm,xk,xai,n,m,time,f,t);
fprintf('Response of a single D.O.F. system under periodic force\n\n');
fprintf('xm   = %10.8e\n',xm);
fprintf('xk   = %10.8e\n',xk);
fprintf('xai  = %10.8e\n',xai);
fprintf('n    = %2.0f\n',n);
fprintf('m    = %2.0f\n',m);
fprintf('time = %10.8e\n\n',time);
```

```
fprintf('Applied force and response: \n\n');
fprintf('   i          t(i)                    f(i)                x(i)')↙
;
fprintf('\n\n');
for i=1:n
   fprintf('  %2.0f    %10.8e      %10.8e      %10.8e\n',i,...
       t(i),f(i),x(i));
end
subplot(121);
plot(t,f);
xlabel('t');
ylabel('F(t)');
%title('F(t)');
subplot(122);
plot(t,x);
%title('x(t)');
xlabel('t');
ylabel('x(t)');

%==========================================================================↙
=====
%
%function period.m
%
%==========================================================================↙
======
function [xsin,xcos,psi,phi,fzero,fc,x,xpc,xps]=period(xm,xk,xai,n,m,time,f↙
,t)
omeg=2.0*3.1416/time;
omegn=sqrt(xk/xm);
sumz=0.0;
for i=1:n
   sumz=sumz+f(i);
end
fzero=2.0*sumz/n;
for j=1:m
   sums=0.0;
   sumc=0.0;
   for i=1:n
      theta=j*omeg*t(i);
      fsin=f(i)*sin(theta);
      fcos=f(i)*cos(theta);
      sums=sums+fsin;
      sumc=sumc+fcos;
   end
   aj=2.0*sumc/n;
   bj=2.0*sums/n;
   r=omeg/omegn;
   phi(j)=atan(2.0*xai*j*r/(1-(j*r)^2));
   con=sqrt((1.0-(j*r)^2)^2+(2.0*xai*j*r)^2);
   xpc(j)=(aj/xk)/con;
   xps(j)=(bj/xk)/con;
end
for i=1:n
```

```
   x(i)=fzero/2/xk;
   for j=1:m
      x(i)=x(i)+xpc(j)*cos(j*omeg*t(i)-phi(j))+xps(j)*sin(j*omeg*t(i)-phi(j↙
));
   end
end
```






```
Results of  Ex4_91
*******************************************
>> program4
xm    = 1.00000000e+000
xk    = 4.00000000e+002
xai   = 1.25000000e-001
n     = 20
m     = 10
time  = 1.00000000e+000

Applied force and response:
```

i	t(i)	f(i)	x(i)
1	5.00000000e-002	6.25000000e+001	3.78789292e-001
2	1.00000000e-001	1.25000000e+002	6.65286858e-001
3	1.50000000e-001	1.87500000e+002	5.00283762e-001
4	2.00000000e-001	2.50000000e+002	2.37031525e-001
5	2.50000000e-001	3.12500000e+002	3.39025654e-001
6	3.00000000e-001	3.75000000e+002	9.42110868e-001
7	3.50000000e-001	4.37500000e+002	1.58854150e+000
8	4.00000000e-001	5.00000000e+002	1.55862192e+000
9	4.50000000e-001	0.00000000e+000	6.56373440e-001
10	5.00000000e-001	0.00000000e+000	-5.23632679e-003

11	5.50000000e-001	0.00000000e+000	-3.77065247e-001
12	6.00000000e-001	0.00000000e+000	-5.06616825e-001
13	6.50000000e-001	0.00000000e+000	-3.57481271e-001
14	7.00000000e-001	0.00000000e+000	4.83338999e-002
15	7.50000000e-001	0.00000000e+000	4.21581779e-001
16	8.00000000e-001	0.00000000e+000	4.69545854e-001
17	8.50000000e-001	0.00000000e+000	1.02656116e-001
18	9.00000000e-001	0.00000000e+000	-3.54840551e-001
19	9.50000000e-001	0.00000000e+000	-5.10135040e-001
20	1.00000000e+000	0.00000000e+000	-1.71815377e-001

4.92

```
%==========================================================================
%
%Program5.m
%Response of a single D.O.F.ystem under arbitrary forcing function
%usingthe methods of section 4.11
%
%==========================================================================
% Run "Program5" in MATLAB command window. Program5.m should be in one
% folder, and set the path to this folder
% following 11 lines contain problem-dependent data
xai=0.1;
omn=31.622777;
xk=1e5;
delt=0.1;
np=21;
np1=20;
np2=19;
%np = number of points at which value of f is known, np1=np-1,np2=np-2
%end of problem-dependent data
xn=xai*omn;
pd=omn*sqrt(1-xai^2);
%solution according to method 1 (using step variation with larger value, Approach 1)
t(1)=0;
f(1) = 0;
ff(1) = 0;
for i=2:np
   t(i)=t(i-1)+delt;
   f(i) = 1000*(1-cos(pi*t(i)));
   ff(i) = 1000*( 1-cos( pi*( t(i)+t(i-1) )/2 ) );
end
for i=1:np1
   delf(i)=f(i+1)-f(i);
end
for j=2:np
   x(j)=0;
   xd(j)=0;
   jm1=j-1;
   for i=1:jm1
      x(j)=x(j)+(delf(i)/xk)*(1-exp(-xn*(t(j)-t(i)))*...
           (cos(pd*(t(j)-t(i)))+(xn/pd)*sin(pd*(t(j)-t(i)))));
      xd(j)=xd(j)+(delf(i)/xk)*exp(-xn*(t(j)-t(i)))*sin(pd*(t(j)-t(i)));
   end
end
for i=2:np
   x1(i)=x(i);
```

```
   xd1(i)=xd(i);
end
%solution according to method 1 (using step variation with smaller value, Approach 2)
for k=2:np2
   delf(k)=delf(k+1);
end
delf(1)=f(3);
delf(np)=f(np);
for j=2:np
   x(j)=0;
   xd(j)=0;
   jm1=j-1;
   for i=1:jm1
      x(j)=x(j)+(delf(i)/xk)*(1-exp(-xn*(t(j)-t(i)))*...
          (cos(pd*(t(j)-t(i)))+(xn/pd)*sin(pd*(t(j)-t(i)))));
      xd(j)=xd(j)+(delf(i)/xk).*exp(-xn*(t(j)-t(i)))*sin(pd*(t(j)-t(i)));
   end
end
for i=2:np
   x2(i)=x(i);
   xd2(i)=xd(i);
end
%solution according to method 2 (using step variation with mid-value, Approach 3)
x(1)=0;
xd(1)=0;
for j=2:np
   del=delt;
   x(j)=(ff(j)/xk)*(1-exp(-xn*del)*(cos(pd*del)+(xn/pd)*sin(pd*del)))...
   +exp(-xn*del)*(x(j-1)*cos(pd*del)+((xd(j-1)+xn*x(j-1))/pd)*sin(pd*del));
   xd(j)=(ff(j)*pd/xk)*exp(-xn*del)*(1+xn^2/(pd^2))*sin(pd*del) + pd* ...
   exp(-xn*del)*(-x(j-1)*sin(pd*del)+((xd(j-1)+xn*x(j-1))/pd)*cos(pd*del)..
.
   -xn*(x(j-1)*cos(pd*del)+((xd(j-1)+xn*x(j-1))/pd)*sin(pd*del))/pd);
end
for i=2:np
   x3(i)=x(i);
   xd3(i)=xd(i);
end
%solution according to method 3 (using linear variation, Approach 4)
x(1)=0;
xd(1)=0;
for j=1:np1
   f(j)=f(j+1);
end
f(np)=0;
for j=2:np
   delf(j)=f(j)-f(j-1);
   x(j)=(delf(j)/(xk*del))*(del-(2*xai/omn)+exp(-xn*del)*((2*xai/omn)...
       *cos(pd*del)-((pd^2-xn^2)/(omn*omn*pd))*sin(pd*del)))+(f(j-1)/xk)...
       *(1-exp(-xn*del)*(cos(pd*del)+(xn/pd)*sin(pd*del)))+exp(-xn*del)*...
       (x(j-1)*cos(pd*del)+((xd(j-1)+xn*x(j-1))/pd)*sin(pd*del));
%  xd(j)=(delf(j)/(xk*del))*(1.-exp(-xn*del)*(((xn^2+pd^2)/(omn^2))*...
%     cos(pd*del)+((xn^3+xn*pd*pd/(pd*(omn^2)))*sin(pd*del)))+(f(j-1)/xk).
```

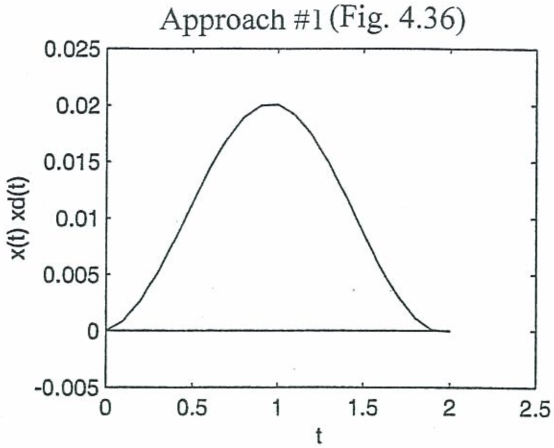
```
   %    *exp(-xn*del)*((xn^2/pd)+pd)*sin(pd*del)+exp(-xn*del)*(xd(j-1)*...
   %   cos(pd*del)-((xn*xd(j-1)+xn*xn*x(j-1)+pd*pd*x(j-1))/pd)*sin(pd*del)))↙
;
   xd(j)=(delf(j)/(xk*del))*(1.-exp(-xn*del)*(cos(pd*del)+xn/pd*...
   sin(pd*del)))+f(j-1)/xk*exp(-xn*del)*omn^2/pd*sin(pd*del)+exp(-xn*del)..↙
.
    *(xd(j-1)*cos(pd*del)-xn/pd*(xd(j-1)+omn/xai*x(j-1))*sin(pd*del));
end
for i=2:np
   x4(i)=x(i);
   xd4(i)=xd(i);
end
fprintf...
('Value     Approach #1         Approach #2         Approach #32        Approach #4\n'↙
);
fprintf...
('   of       (Fig. 4.36)         (Fig. 4.36)         (Fig. 4.36)         (Fig. 4.36)  ');
fprintf('\n\n');
fprintf...
(' I       x(i)              x(i)              x(i)              x(i)\n\n');
for i=2:np
   fprintf...
   ('%2.0f  %8.6e   %8.6e   %8.6e    %8.6e\n',i,x1(i),x2(i),x3(i),...
   x4(i));
end
fprintf...
('\n\n I     xd(i)             xd(i)             xd(i)             xd(i)');
fprintf('\n\n');
for i=2:np
    fprintf...
    ('%2.0f  %8.6e   %8.6e   %8.6e    %8.6e\n',i,xd1(i),xd2(i),...
    xd3(i),xd4(i));
end
subplot(221);
plot(t,x1);
hold on;
plot(t,xd1);
xlabel('t');
ylabel('x(t) xd(t)');
title('Method #1 (Fig. 4.36)');
subplot(222);
plot(t,x2);
hold on;
plot(t,xd2);
xlabel('t');
ylabel('x(t) xd(t)');
title('Method #1 (Fig. 4.36)');
subplot(223);
plot(t,x3);
hold on;
plot(t,xd3);
xlabel('t');
ylabel('x(t) xd(t)');
title('Method #2 (Fig. 4.36)');
subplot(224);
```

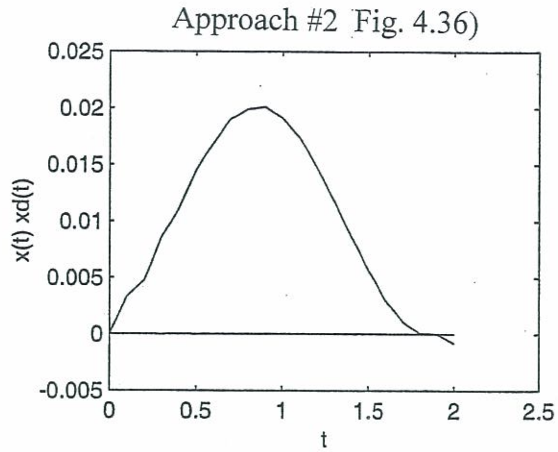
```
plot(t,x4);
hold on;
plot(t,xd4);
xlabel('t');
ylabel('x(t) xd(t)');
title('Method #3 (Fig. 4.36) ');
```

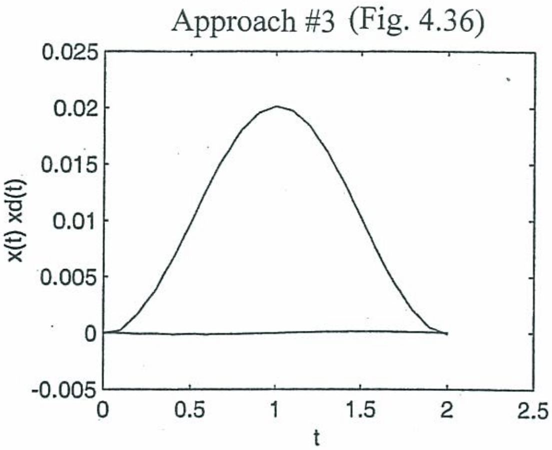
```
Results of Ex4_92
*************************************
>> program5
```

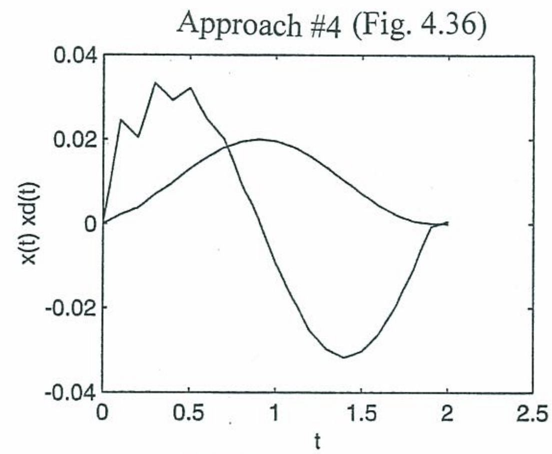
Value of I	Approach #1 (Fig. 4.36) x(i)	Approach #2 (Fig. 4.36) x(i)	Approach #3 (Fig. 4.36) x(i)	Approach #4 (Fig. 4.36) x(i)
2	8.463498e-004	3.302553e-003	2.128980e-004	2.112992e-003
3	2.685367e-003	4.719852e-003	1.729506e-003	3.731773e-003
4	5.169912e-003	8.506840e-003	3.803611e-003	6.890461e-003
5	8.178657e-003	1.108890e-002	6.668076e-003	9.675896e-003
6	1.132821e-002	1.454957e-002	9.724658e-003	1.298875e-002
7	1.437507e-002	1.684652e-002	1.290593e-002	1.564646e-002
8	1.697375e-002	1.899708e-002	1.573148e-002	1.801666e-002
9	1.890432e-002	1.988503e-002	1.804796e-002	1.940816e-002
10	1.995267e-002	2.008383e-002	1.953876e-002	2.002060e-002
11	2.003452e-002	1.909252e-002	2.012346e-002	1.954838e-002
12	1.912848e-002	1.735920e-002	1.969707e-002	1.821740e-002
13	1.733299e-002	1.479759e-002	1.833615e-002	1.602653e-002
14	1.481670e-002	1.184504e-002	1.614852e-002	1.328634e-002
15	1.183110e-002	8.654518e-003	1.336683e-002	1.019478e-002
16	8.664674e-003	5.637520e-003	1.024989e-002	7.105849e-003
17	5.630116e-003	3.017069e-003	7.112632e-003	4.284352e-003
18	3.022466e-003	1.102377e-003	4.254988e-003	2.033946e-003
19	1.098443e-003	4.244805e-005	1.961909e-003	5.548287e-004
20	4.531599e-005	-3.095846e-005	4.540486e-004	6.474018e-006
21	-3.304897e-005	-8.237715e-004	-1.182168e-004	-4.635301e-006

I	xd(i)	xd(i)	xd(i)	xd(i)
2	-1.724478e-006	-6.729107e-006	-1.378674e-005	2.450722e-002
3	-2.490738e-006	2.014596e-006	-8.816757e-005	2.042474e-002
4	-3.247781e-006	-9.183878e-006	-7.008044e-005	3.328193e-002
5	-3.764329e-006	1.429730e-006	-1.344394e-004	2.918735e-002
6	-3.674900e-006	-8.092879e-006	-9.999247e-005	3.213625e-002
7	-3.530832e-006	1.215821e-006	-1.331627e-004	2.478209e-002
8	-2.722577e-006	-5.267651e-006	-8.596171e-005	2.017583e-002
9	-1.950123e-006	2.028444e-006	-8.738285e-005	9.855085e-003
10	-7.153334e-007	-1.882855e-006	-3.287885e-005	1.272298e-003
11	3.543858e-007	3.391564e-006	-1.391042e-005	-9.388100e-003
12	1.587911e-006	1.060844e-006	3.774625e-005	-1.771827e-002
13	2.501541e-006	4.446538e-006	6.063053e-005	-2.533370e-002
14	3.304589e-006	2.776498e-006	9.749376e-005	-2.973266e-002
15	3.675776e-006	4.478047e-006	1.091071e-004	-3.175330e-002
16	3.773808e-006	2.884849e-006	1.223568e-004	-3.028136e-002
17	3.433694e-006	3.237571e-006	1.140193e-004	-2.612284e-002
18	2.811634e-006	1.542582e-006	1.019865e-004	-1.920680e-002
19	1.871907e-006	1.035826e-006	7.419301e-005	-1.055539e-002
20	7.820400e-007	-6.050694e-007	4.359287e-005	-7.662515e-004
21	-4.100735e-007	2.056210e-006	5.299422e-006	5.589628e-004





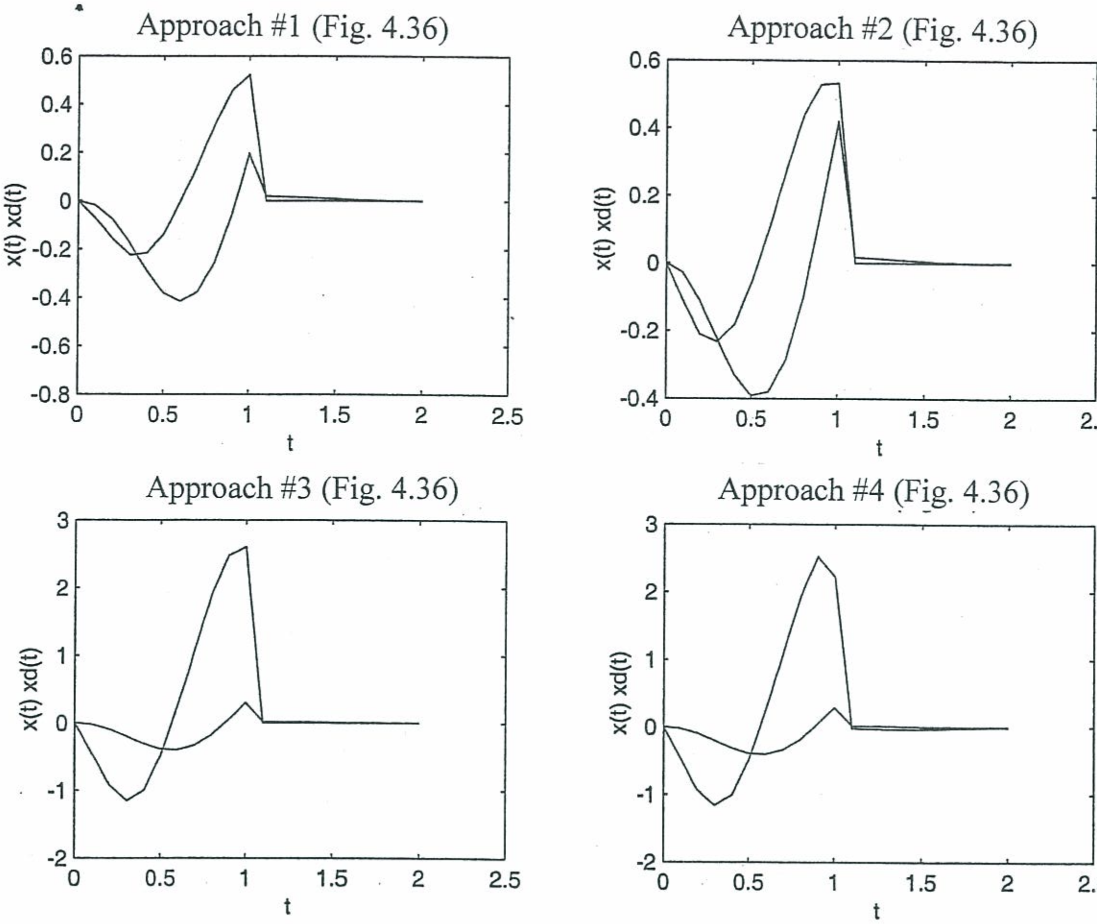




4.93

```
%=========================================================================
%
%Program5.m
%Response of a single D.O.F.ystem under arbitrary forcing function
%usingthe methods of section 4.8
%
%=========================================================================
% Run "Program5" in MATLAB command window. Program5.m should be in one
% folder, and set the path to this folder
% following 11 lines contain problem-dependent data
xai=0.1;
omn=5;
xk=50;
delt=0.1;
np=11;
np1=10;
np2=9;
%np = number of points at which value of f is known, np1=np-1,np2=np-2
%end of problem-dependent data
xn=xai*omn;
pd=omn*sqrt(1-xai^2);
%solution according to method 1 (using step variation with larger value, Approach 1)
t(1)=0;
for i=2:np
   t(i)=t(i-1)+delt;
end
```

```
f=[0.0 -8.0 -12.0 -15.0 -13.0 -11.0 -7.0 -4.0 3.0 10.0 15.0];
ff=[-8.0 -10.0 -13.5 -14.0 -12.0 -9.0 -5.5 -0.5 6.5 12.5 16.5];
for i=1:np1
   delf(i)=f(i+1)-f(i);
end
     :
```




```
Results of Ex4_93
******************************
>> program5
```

Value of I	Approach #1 (Fig. 4.36) x(i)	Approach #2 (Fig. 4.36) x(i)	Approach #3 (Fig. 4.36) x(i)	Approach #4 (Fig. 4.36) x(i)
2	-1.895258e-002	-2.842886e-002	-2.369072e-002	-2.216415e-002
3	-7.844079e-002	-1.105540e-001	-9.449737e-002	-9.232748e-002
4	-1.770172e-001	-2.242651e-001	-2.006411e-001	-2.003034e-001
5	-2.901286e-001	-3.307430e-001	-3.104358e-001	-3.118343e-001
6	-3.807183e-001	-3.918096e-001	-3.862639e-001	-3.896705e-001
7	-4.157648e-001	-3.812882e-001	-3.985265e-001	-4.025740e-001
8	-3.761180e-001	-2.849867e-001	-3.305524e-001	-3.357629e-001
9	-2.546674e-001	-1.008869e-001	-1.777772e-001	-1.828250e-001
10	-5.487595e-002	1.476431e-001	4.638356e-002	4.342363e-002
11	1.971323e-001	4.198656e-001	3.061299e-001	2.920608e-001

I	xd(i)	xd(i)	xd(i)	xd(i)
2	-7.263201e-002	-1.089480e-001	-4.562370e-001	-4.597525e-001

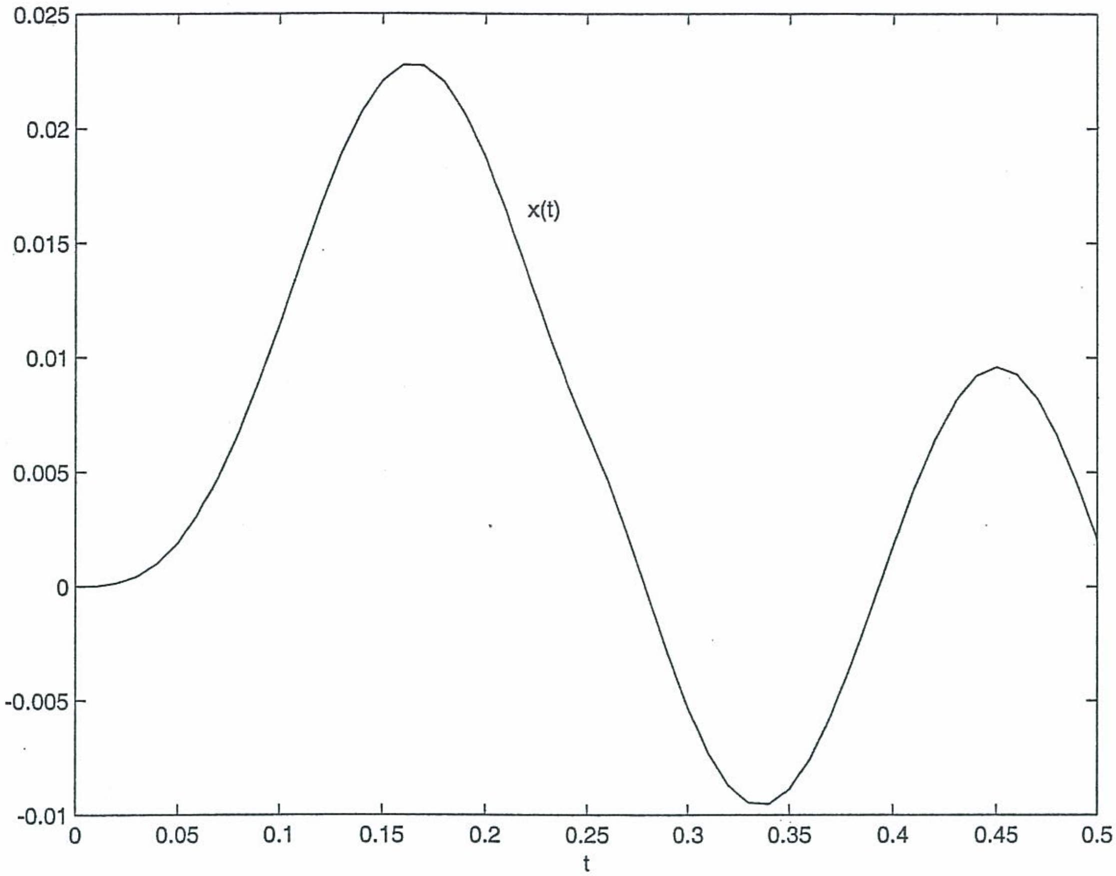
3	-1.577455e-001	-2.093812e-001	-9.224406e-001	-9.313379e-001
4	-2.252429e-001	-2.333148e-001	-1.152170e+000	-1.162393e+000
5	-2.156793e-001	-1.824523e-001	-1.000343e+000	-1.007436e+000
6	-1.386163e-001	-5.760411e-002	-4.930222e-001	-4.919218e-001
7	-2.742288e-004	9.602153e-002	2.405741e-001	2.518514e-001
8	1.522037e-001	2.762085e-001	1.076426e+000	1.100724e+000
9	3.182622e-001	4.384471e-001	1.901304e+000	1.938544e+000
10	4.579186e-001	5.284869e-001	2.478437e+000	2.523778e+000
11	5.229886e-001	5.322201e-001	2.605688e+000	2.224476e+000

4.94

```
% Ex4_94.m
% This program will use the function dfunc4_94.m , they should
% be in the same folder
tspan = [0: 0.01: 0.5];
x0 = [0.0; 0.0];
[t,x] = ode23('dfunc4_94', tspan, x0);
disp('     t          x(t)     xd(t)');
disp([t x]);
plot(t,x(:,1));
xlabel('t');
gtext('x(t)');
```

```
% dfunc4_94.m
function f = dfunc4_94(t,x)
F = 200*t - 200*t*stepfun(t,0.1)...
      +20*stepfun(t,0.1)-20*stepfun(t,0.25);
f = zeros(2,1);
f(1) = x(2);
f(2) = F/2 - 1500 * x(1)/2;
```

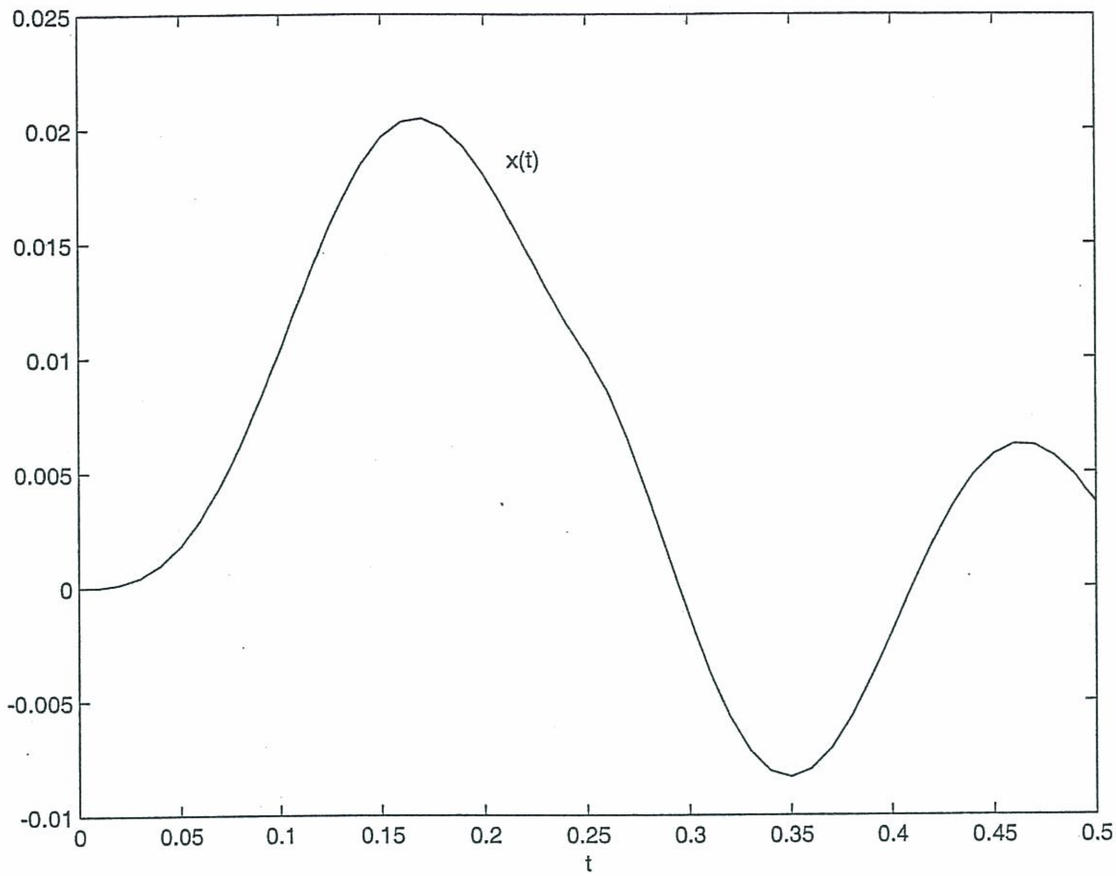
```
Results of Ex4_94
*********************************
>> Ex4_94.m
      t          x(t)      xd(t)
         0          0          0
    0.0100     0.0000     0.0050
    0.0200     0.0001     0.0195
    0.0300     0.0004     0.0425
    0.0400     0.0010     0.0723
    0.0500     0.0019     0.1067
    0.0600     0.0031     0.1430
    .
    .
    .
    0.4100     0.0043     0.2352
    0.4200     0.0064     0.1949
    0.4300     0.0081     0.1400
    0.4400     0.0092     0.0748
    0.4500     0.0096     0.0039
    0.4600     0.0093    -0.0671
    0.4700     0.0083    -0.1332
    0.4800     0.0066    -0.1893
    0.4900     0.0045    -0.2313
    0.5000     0.0021    -0.2560
```



4.95

```
% Ex4_95.m
% This program will use the function dfunc4_95.m , they should
% be in the same folder
tspan = [0: 0.01: 0.5];
x0 = [0.0; 0.0];
[t,x] = ode23('dfunc4_95 ', tspan, x0);
disp('       t          x(t)      xd(t)');
disp([t x]);
plot(t,x(:,1));
xlabel('t');
gtext('x(t)');
```

```
% dfunc4_95.m
function f = dfunc4_95(t,x)
F = 200*t - 200*t*stepfun(t,0.1)...
        +20*stepfun(t,0.1)-20*stepfun(t,0.25);
f = zeros(2,1);
f(1) = x(2);
f(2) = F/2 - 10 * x(2)/2 - 1500 * x(1)/2;
```




```
Results of Ex4_95
**********************************
>> Ex4_95
```

t	x(t)	xd(t)
0	0	0
0.0100	0.0000	0.0049
0.0200	0.0001	0.0189
0.0300	0.0004	0.0405
0.0400	0.0010	0.0678
0.0500	0.0018	0.0986
0.0600	0.0029	0.1303
0.0700	0.0044	0.1608
0.0800	0.0061	0.1877
0.0900	0.0081	0.2093
0.1000	0.0103	0.2244
0.4100	-0.0000	0.1962
0.4200	0.0019	0.1797
0.4300	0.0035	0.1510
0.4400	0.0049	0.1127
0.4500	0.0058	0.0680
0.4600	0.0062	0.0206
0.4700	0.0062	-0.0259
0.4800	0.0057	-0.0683
0.4900	0.0048	-0.1037
0.5000	0.0037	-0.1298

4.96

$$x(t) = \frac{1}{m\,\omega_d} \int_0^t F(\tau)\, e^{-\zeta\omega_n (t-\tau)} \sin \omega_d (t-\tau)\, d\tau$$

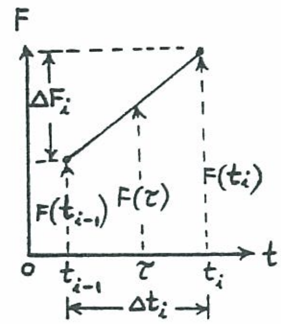
But $\sin \omega_d (t-\tau) = \sin \omega_d t \cos \omega_d \tau - \cos \omega_d t \sin \omega_d \tau$

$$x(t) = \left\{ A(t) \cdot \sin \omega_d t - B(t) \cdot \cos \omega_d t \right\} \frac{e^{-\zeta\omega_n t}}{m\,\omega_d}$$

where

$$A(t) = \int_0^t F(\tau)\, e^{\zeta\omega_n \tau} \cos \omega_d \tau \cdot d\tau$$

$$B(t) = \int_0^t F(\tau)\, e^{\zeta\omega_n \tau} \sin \omega_d \tau \cdot d\tau$$



Let $F(\tau)$ be taken as a piecewise linear function during (t_{i-1}, t_i) as

$$F(\tau) = F_{i-1} + \left(\frac{\tau - t_{i-1}}{t_i - t_{i-1}}\right)\left(F_i - F_{i-1}\right)$$

We can write

$$A(t_i) = A(t_{i-1}) + \int_{t_{i-1}}^{t_i} \left(\frac{\tau}{\Delta t_i} \Delta F_i\right) e^{\zeta \omega_n \tau} \cos \omega_d \tau \cdot d\tau$$

$$+ \int_{t_{i-1}}^{t_i} \left(F_{i-1} - t_{i-1} \frac{\Delta F_i}{\Delta t_i}\right) e^{\zeta \omega_n \tau} \cos \omega_d \tau \cdot d\tau$$

$$= A(t_{i-1}) + \frac{\Delta F_i}{\Delta t_i} P_1 + \left(F_{i-1} - t_{i-1} \frac{\Delta F_i}{\Delta t_i}\right) P_2$$

where

$$P_1 = \int_{t_{i-1}}^{t_i} \tau \, e^{\zeta \omega_n \tau} \cos \omega_d \tau \cdot d\tau$$

$$P_2 = \int_{t_{i-1}}^{t_i} e^{\zeta \omega_n \tau} \cos \omega_d \tau \cdot d\tau$$

$$B(t_i) = B(t_{i-1}) + \int_{t_{i-1}}^{t_i} \left(\frac{\tau}{\Delta t_i} \Delta F_i\right) e^{\zeta \omega_n \tau} \sin \omega_d \tau \cdot d\tau$$

$$+ \int_{t_{i-1}}^{t_i} \left(F_{i-1} - t_{i-1} \frac{\Delta F_i}{\Delta t_i}\right) e^{\zeta \omega_n \tau} \sin \omega_d \tau \cdot d\tau$$

$$= B(t_{i-1}) + \frac{\Delta F_i}{\Delta t_i} P_4 + \left(F_{i-1} - t_{i-1} \frac{\Delta F_i}{\Delta t_i}\right) P_3$$

where

$$P_3 = \int_{t_{i-1}}^{t_i} e^{\zeta \omega_n \tau} \sin \omega_d \tau \cdot d\tau$$

$$P_4 = \int_{t_{i-1}}^{t_i} \tau \, e^{\zeta \omega_n \tau} \sin \omega_d \tau \cdot d\tau$$

Assuming $A(t_1 = 0) = B(t_1 = 0) = 0$,

$$x(t_i) = \frac{e^{-\zeta \omega_n t_i}}{m \omega_d} \left[A(t_i) \sin \omega_d t_i - B(t_i) \cos \omega_d t_i \right]$$

The integrals in P_1, P_2, P_3 and P_4 can be evaluated in closed form.

The computer program and output are given.

```
C ==================================================================
C
C PROBLEM 4.96
C NUMERICAL INTEGRATION OF DUHAMEL INTEGRAL
C
C ==================================================================
C PROBLEM-DEPENDENT DATA
      DIMENSION F(21),X(21),DELT(21),T(21),A(21),B(21)
      NP=21
      XAI=0.1
      OMN=1.0
```

```
      XM=1.0
      DATA F/1.0,.8436,.6910,.5460,.4122,.2929,.1910,.1090,
     2 .04894,.01231,.0,.0,.0,.0,.0,.0,.0,.0,.0,.0,.0/
      DO 10 I=1,21
 10   DELT(I)=0.31416
C END OF PROBLEM-DEPENDENT DATA
      A(1)=0.0
      B(1)=0.0
      OMD=OMN*SQRT(1.0-XAI**2)
      T(1)=0.0
      DO 20 I=2,NP
      T(I)=T(I-1)+DELT(I)
      TIME=T(I)
      CALL PI1(TIME,XAI,OMN,OMD,PP1)
      CALL PI2(TIME,XAI,OMN,OMD,PP2)
      CALL PI3(TIME,XAI,OMN,OMD,PP3)
      CALL PI4(TIME,XAI,OMN,OMD,PP4)
      TIME=T(I-1)
      CALL PI1(TIME,XAI,OMN,OMD,PM1)
      CALL PI2(TIME,XAI,OMN,OMD,PM2)
      CALL PI3(TIME,XAI,OMN,OMD,PM3)
      CALL PI4(TIME,XAI,OMN,OMD,PM4)
      P1=PP1-PM1
      P2=PP2-PM2
      P3=PP3-PM3
      P4=PP4-PM4
      DELF=F(I)-F(I-1)
      A(I)=A(I-1)+(DELF/DELT(I))*P1+(F(I-1)-T(I-1)*DELF/DELT(I))*P2
      B(I)=B(I-1)+(DELF/DELT(I))*P4+(F(I-1)-T(I-1)*DELF/DELT(I))*P3
      X(I)=(EXP(-XAI*OMN*T(I))/(XM*OMD))*(A(I)*SIN(OMD*T(I))-
     2 B(I)*COS(OMD*T(I)))
 20   CONTINUE
      PRINT 30
 30   FORMAT (//,2X,41H NUMERICAL EVALUATION OF DUHAMEL INTEGRAL,
     2 //,5X,2H I,6X,5H T(I),10X,5H F(I),10X,5H X(I),/)
      DO 40 I=2,NP
 40   PRINT 50, I,T(I),F(I),X(I)
 50   FORMAT (2X,I5,3E15.8)
      STOP
      END
C ==================================================================
C
C SUBROUTINE PI1
C
C ==================================================================
      SUBROUTINE PI1 (T,XAI,OMN,OMD,P)
      DEN=(XAI*OMN)**2+OMD**2
      P=(T*EXP(XAI*OMN*T)/DEN)*(XAI*OMN*COS(OMD*T)+OMD*SIN(OMD*T))
     2 -(EXP(XAI*OMN*T)/(DEN**2))*(((XAI*OMN)**2-OMD**2)*COS(OMD*T)
     3 +2.0*XAI*OMN*OMD*SIN(OMD*T))
      RETURN
      END
```

```
C ==================================================================
C
C SUBROUTINE PI2
C
C ==================================================================
      SUBROUTINE PI2 (T,XAI,OMN,OMD,P)
      DEN=(XAI*OMN)**2+OMD**2
      P=(EXP(XAI*OMN*T)/DEN)*(XAI*OMN*COS(OMD*T)+OMD*SIN(OMD*T))
      RETURN
      END
C ==================================================================
C
C SUBROUTINE PI3
C
C ==================================================================
      SUBROUTINE PI3 (T,XAI,OMN,OMD,P)
      DEN=(XAI*OMN)**2+OMD**2
      P=(EXP(XAI*OMN*T)/DEN)*(XAI*OMN*SIN(OMD*T)-OMD*COS(OMD*T))
      RETURN
      END
C ==================================================================
C
C SUBROUTINE PI4
C
C ==================================================================
      SUBROUTINE PI4 (T,XAI,OMN,OMD,P)
      DEN=(XAI*OMN)**2+OMD**2
      P=(T*EXP(XAI*OMN*T)/DEN)*(XAI*OMN*SIN(OMD*T)-OMD*COS(OMD*T))
     2 -(EXP(XAI*OMN*T)/(DEN**2))*(((XAI*OMN)**2-OMD**2)*SIN(OMD*T)
     3 -2.0*XAI*OMN*OMD*COS(OMD*T))
      RETURN
      END
```

NUMERICAL EVALUATION OF DUHAMEL INTEGRAL

I	T(I)	F(I)	X(I)
2	0.31415999E+00	0.84359998E+00	0.45415938E-01
3	0.62831998E+00	0.69099998E+00	0.16377741E+00
4	0.94247997E+00	0.54600000E+00	0.32499743E+00
5	0.12566404E+01	0.41219997E+00	0.49747676E+00
6	0.15708008E+01	0.29290003E+00	0.65152758E+00
7	0.18849611E+01	0.19099998E+00	0.76239210E+00
8	0.21991215E+01	0.10900003E+00	0.81256396E+00
9	0.25132818E+01	0.48939999E-01	0.79324198E+00
10	0.28274422E+01	0.12309998E-01	0.70482814E+00
11	0.31416025E+01	0.00000000E+00	0.55646867E+00
12	0.34557629E+01	0.00000000E+00	0.36454141E+00
13	0.37699232E+01	0.00000000E+00	0.14971566E+00
14	0.40840836E+01	0.00000000E+00	-0.66231847E-01
15	0.43982439E+01	0.00000000E+00	-0.26274461E+00
16	0.47124043E+01	0.00000000E+00	-0.42236131E+00
17	0.50265646E+01	0.00000000E+00	-0.53218621E+00
18	0.53407249E+01	0.00000000E+00	-0.58483124E+00
19	0.56548853E+01	0.00000000E+00	-0.57878375E+00
20	0.59690456E+01	0.00000000E+00	-0.51819199E+00
21	0.62832060E+01	0.00000000E+00	-0.41212624E+00

4.97 The computer program of problem 4.96 can be used to find the relative displacement $z(t)$ of the water tank provided $-m\ddot{y}(\tau)$ is used in place of $F(\tau)$.

Here $\zeta = 0$, $\omega_n = 22.3607$ rad/sec and

$$F(\tau) = -10000 \times 9.81 \times \ddot{y}(\tau) \quad \text{if } \ddot{y} \text{ is in } g\text{'s}.$$

The problem-dependent data for the program of problem 4.96 and the output are given below.

```
C ==============================================================
C
C PROBLEM 4.97
C
C ==============================================================
C PROBLEM-DEPENDENT DATA
      DIMENSION F(15),X(15),DELT(15),T(15),A(15),B(15)
      NP=15
      XAI=0.0
      OMN=22.3607
      XM=10000.0
      DATA F/.0,.45,-.8,-.9,-.6,-.75,-.7,.55,1.75,1.65,.25,
     2 -1.1,-1.4,-1.05,.0/
      DO 10 I=1,15
 10   DELT(I)=0.025
      DO 11 I=1,15
 11   F(I)=-XM*9.81*F(I)
C END OF PROBLEM-DEPENDENT DATA


NUMERICAL EVALUATION OF DUHAMEL INTEGRAL

    I        T(I)              F(I)             X(I)

    2 0.24999999E-01-0.44144996E+05-0.45271125E-03
    3 0.49999997E-01 0.78480000E+05-0.17453337E-02
    4 0.74999988E-01 0.88290000E+05 0.11150776E-02
    5 0.99999964E-01 0.58860004E+05 0.86095147E-02
    6 0.12499994E+00 0.73575000E+05 0.17519452E-01
    7 0.14999992E+00 0.68670000E+05 0.25374368E-01
    8 0.17499989E+00-0.53955000E+05 0.28478131E-01
    9 0.19999987E+00-0.17167500E+06 0.19676924E-01
   10 0.22499985E+00-0.16186494E+06-0.42601228E-02
   11 0.24999982E+00-0.24525000E+05-0.35448216E-01
   12 0.27499980E+00 0.10791006E+06-0.57387743E-01
   13 0.29999977E+00 0.13734000E+06-0.56341648E-01
   14 0.32499975E+00 0.10300506E+06-0.30433871E-01
   15 0.34999973E+00 0.00000000E+00 0.10307007E-01
```

4.98 The problem-dependent data (to be used in the program of Problem 4.96) and output are given.

```
C PROBLEM-DEPENDENT DATA
      DIMENSION F(30),X(30),DELT(30),T(30),A(30),B(30)
      NP=30
      XAI=0.0
      OMN=8.660254
      XM=2.0
      DATA F /60.0,60.0,60.0,60.0,60.0,100.0,100.0,100.0,100.0,100.0,
     2 30.0,30.0,30.0,30.0,30.0,30.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,
     3 0.0,0.0,0.0,0.0,0.0,0.0/
      DO 10 I=1,30
 10   DELT(I)=0.01
C END OF PROBLEM-DEPENDANT DATA
```

NUMERICAL EVALUATION OF DUHAMEL INTEGRAL

I	T(I)	F(I)	X(I)
2	0.99999998E-02	0.60000000E+02	0.14990796E-02
3	0.20000000E-01	0.60000000E+02	0.59849964E-02
4	0.29999999E-01	0.60000000E+02	0.13424230E-01
5	0.39999999E-01	0.60000000E+02	0.23760952E-01
6	0.50000001E-01	0.10000000E+03	0.37250735E-01
7	0.60000002E-01	0.10000000E+03	0.55125091E-01
8	0.70000000E-01	0.10000000E+03	0.77583194E-01
9	0.79999998E-01	0.10000000E+03	0.10445669E+00
10	0.89999996E-01	0.10000000E+03	0.13554408E+00
11	0.99999994E-01	0.30000000E+02	0.17002963E+00
12	0.10999999E+00	0.30000000E+02	0.20532255E+00
13	0.11999999E+00	0.30000000E+02	0.24057558E+00
14	0.13000000E+00	0.30000000E+02	0.27552453E+00
15	0.14000000E+00	0.30000000E+02	0.30990744E+00
16	0.15000001E+00	0.30000000E+02	0.34346649E+00
17	0.16000001E+00	0.00000000E+00	0.37570029E+00
18	0.17000002E+00	0.00000000E+00	0.40536803E+00
19	0.18000002E+00	0.00000000E+00	0.43199742E+00
20	0.19000003E+00	0.00000000E+00	0.45538884E+00
21	0.20000003E+00	0.00000000E+00	0.47536695E+00
22	0.21000004E+00	0.00000000E+00	0.49178207E+00
23	0.22000004E+00	0.00000000E+00	0.50451112E+00
24	0.23000005E+00	0.00000000E+00	0.51345861E+00
25	0.24000005E+00	0.00000000E+00	0.51855767E+00
26	0.25000006E+00	0.00000000E+00	0.51976985E+00
27	0.26000005E+00	0.00000000E+00	0.51708633E+00
28	0.27000004E+00	0.00000000E+00	0.51052707E+00
29	0.28000003E+00	0.00000000E+00	0.50014120E+00
30	0.29000002E+00	0.00000000E+00	0.48600671E+00

4.99

(i) Find natural frequency:

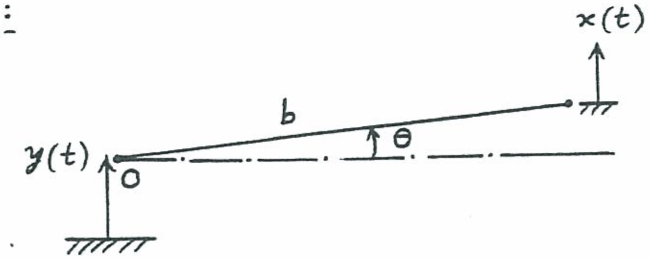
$$J_0 = m b^2 = 9 m a^2$$

$$\frac{1}{2} k_t \theta^2 = \frac{1}{2} k (a\theta)^2 = \frac{1}{2} k a^2 \theta^2 \Rightarrow k_t = k a^2$$

$$\omega_n = \sqrt{\frac{k_t}{J_0}} = \sqrt{\frac{k a^2}{9 m a^2}} = \frac{1}{3}\sqrt{\frac{k}{m}} = 2\pi(10) = 20\pi \ \frac{\text{rad}}{\text{sec}} \qquad (E_1)$$

(ii) Find equation of motion:

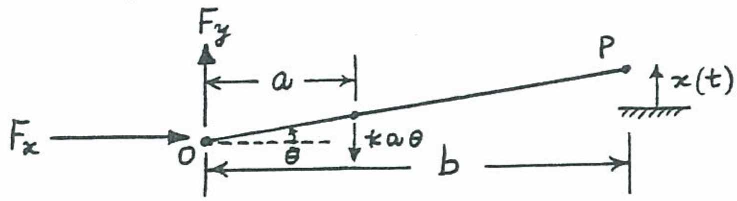
When the base and hence the pivot O is displaced by $y(t)$, the displacement of mass m, $x(t)$, is given by



$$x(t) = y(t) + b\,\theta(t) \qquad (E_2)$$

Relative displacement of mass is

$$z(t) = x(t) - y(t) = b\,\theta(t) \qquad (E_3)$$



Equation of motion:

$\sum$ Forces along y (vertical) direction $= 0 \Rightarrow F_y - k a \theta = m\ddot{x}$ (E_4)

$\sum$ Moments about P $= 0$ $\Rightarrow F_y \cdot b - k a \theta (b - a) = 0$ (E_5)

Solve (E_4) for F_y and substitute the result in (E_5):

$$(k a \theta + m\ddot{x})\,b - k a \theta (b - a) = 0$$

i.e., $$m b \ddot{x} + \theta k a^2 = 0 \qquad (E_6)$$

But $x = y + z$ and $\theta(t) = \frac{1}{b} z(t)$

Hence (E_6) becomes

$$m b \ddot{z} + \frac{k a^2}{b} z = -m b \ddot{y}$$

or $$m\ddot{z} + \frac{k a^2}{b^2} z = -m\ddot{y} \qquad (E_7)$$

Eq. (E_7) can be compared with a standard forced vibration equation for an undamped system:

$$\underset{\sim}{m}\ \ddot{x} + \underset{\sim}{k}\ x = \underset{\sim}{F}(t) \qquad (E_8)$$

Comparison of (E_7) and (E_8) shows that

$$x = z, \quad \underset{\sim}{m} = m, \quad \underset{\sim}{k} = \frac{k a^2}{b^2} \quad \text{and} \quad \underset{\sim}{F} = -m\ \ddot{y} \qquad (E_9)$$

(iii) Find solution:

Solution of Eq. (E_8) under a rectangular pulse is given in problem 4.20. Hence the solution of problem 4.20 can be used to find the relative displacement.

(iv) Design:

Eq. (E_1) gives $\sqrt{k} = 60\pi\sqrt{m}$ or $k = 3600\pi^2 m$ (E_{10})

Following procedure can be used to solve the problem:

a. Assume a value of a.
b. Assume a small value of m.
c. Find k from Eq. (E_{10}).
d. Evaluate the solution numerically as outlined in part (iii).
e. If the maximum relative displacement is larger than or equal to 0.02 m, the design is complete.
f. Otherwise, increase the value of m and go to step c.
g. If necessary change the value of a in step a.

4.100 Let k and m denote the equivalent stiffness and mass of the cutting head.

Equation of motion is: $m\ddot{x} + kx = F(t)$ (E_1)

where $F(t)$ is given by Figs. 4.75(a) and 4.75(b).

Solution of Eq. (E_1) under the force given by Figs. 4.75(a) can be obtained as in problem 4.22. Solution of (E_1) under the force of Fig. (4.75(b)) can be determined using a procedure similar to that of problem 4.25.

The values of k and m can be determined as follows:

a. Assume a small value for m.
b. Assume a small value for k.
c. Evaluate the response under $F(t)$ given by Fig. 4.75(a).

d. Evaluate the response under $F(t)$ given by Fig. (4.75(b).

e. If the responses in steps c and d are approximately equal to 0.1 mm and 0.05 mm, the current values of m and k are the desired values.

f. Otherwise, increment the value of m and/or k and go to step c.

4.101 Model the system as a single d.o.f. torsional system with:

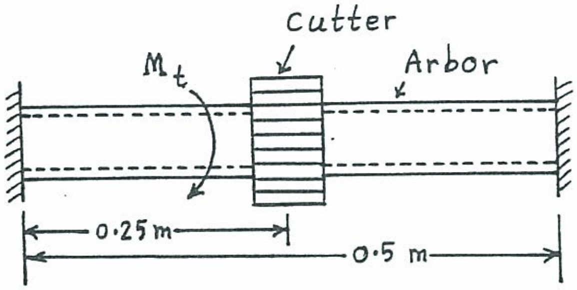
$J_0 = 0.1$ N–m^2 ; $c_t = 0$ (no damping assumed) ;

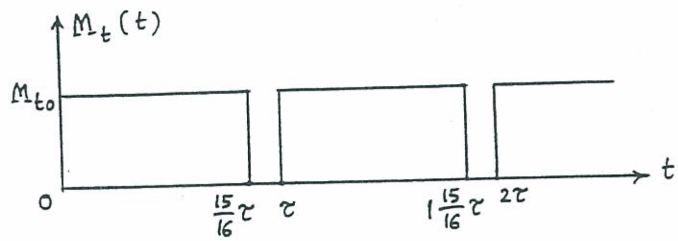
$$k_t = \frac{G\,J}{\ell} = \frac{1}{0.25}\,(80\,(10^9))\left(\frac{\pi}{32}\,(d_o^4 - d_i^4)\right)$$

where d_o = outer diameter of shaft, and d_i = inner diameter.

$$k_t = 62.832\,(10^9)\,(d_o^4 - d_i^4) \tag{1}$$

Torque acting on the arbor due to breakage of one tooth can be modeled as shown in figure.





M_{t0} = 500 N-m, $\tau = \dfrac{2\,\pi}{\omega} = \dfrac{2\,\pi}{N\,(2\,\pi)/60} = 60/N = 60/1000 = 0.06$ sec where N = speed of cutter = 1000 rpm. Express $M_t(t)$ in Fourier series:

$$M_t(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos n\,\omega\,t + b_n \sin n\,\omega\,t\right) \tag{2}$$

(see solution of Problem 4.6 for procedure). where $\omega = \dfrac{2\,\pi\,N}{60} = 104.72$ rad/sec.

Equation of motion:

$$J_0\,\ddot{\theta} + k_t\,\theta = M_t(t) \tag{3}$$

where $M_t(t)$ is given by Eq. (2). Find solution of Eq. (3) and determine the maximum value of θ, θ_{max}. This value must be less than 1°.

Note:
The solution requires an iterative procedure. Assume values for d_i and d_o. Compute k_t, find solution of $\theta(t)$, and the value of θ_{max}. If θ_{max} exceeds 1°, choose a different set of values for d_i and d_o and continue the process until θ_{max} comes out to be less than 1°.
