ESPOL

ECUACIONES DIFERENCIALES (2DO PARCIAL)

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[ERICK CONDE]

RESOLUCIÓN DE ECUACIONES DIFERENCIALES ALREDEDOR DE PUNTOS SINGULARES

MÉTODO DE FROBENIUS

1) $\boldsymbol{xy'' - y' + 4x^3y = 0}$

$$\lim_{x\to 0} x\,\frac{q(x)}{p(x)} \quad\Rightarrow\quad \lim_{x\to 0} x\,\frac{-1}{x} = -1$$

$$\lim_{x\to 0} x^2\,\frac{r(x)}{p(x)} \quad\Rightarrow\quad \lim_{x\to 0} x^2\,\frac{4x^3}{x} = 0$$

$$y(x) = \sum_{n=0}^{+\infty} a_n\, x^{n+r}$$

$$y'(x) = \sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r-2}$$

$$x\sum_{n=0}^{+\infty}(n+r)(n+r-1)a_n\, x^{n+r-2} - \sum_{n=0}^{+\infty}(n+r)a_n\, x^{n+r-1} + 4x^3\sum_{n=0}^{+\infty} a_n\, x^{n+r} = 0$$

$$\sum_{n=0}^{+\infty}(n+r)(n+r-1)a_n\, x^{n+r-1} - \sum_{n=0}^{+\infty}(n+r)a_n\, x^{n+r-1} + 4\sum_{n=0}^{+\infty} a_n\, x^{n+r+3} = 0$$

Multiplicando por "x" a toda la expresión:

$$x\sum_{n=0}^{+\infty}(n+r)(n+r-1)a_n\, x^{n+r-1} - x\sum_{n=0}^{+\infty}(n+r)a_n\, x^{n+r-1} + 4x\sum_{n=0}^{+\infty} a_n\, x^{n+r+3} = 0$$

$$\sum_{n=0}^{+\infty}(n+r)(n+r-1)a_n\, x^{n+r} - \sum_{n=0}^{+\infty}(n+r)a_n\, x^{n+r} + 4\sum_{n=0}^{+\infty} a_n\, x^{n+r+4} = 0$$

$$M = n+4$$

$$\sum_{n=0}^{+\infty}(n+r)(n+r-1)a_n\, x^{n+r} - \sum_{n=0}^{+\infty}(n+r)a_n\, x^{n+r} + 4\sum_{M=4}^{+\infty} a_{M-4}\, x^{M+r} = 0$$

Generando términos hasta n=4

$$a_0(r)(r-1)x^r + a_1(r+1)(r)x^{r+1} + a_2(r+2)(R+1)x^{r+2} + a_3(r+3)(r+2)x^{r+3} -$$

$$a_0(r)x^r - a_1(r+1)x^{r+1} - a_2(r+2)x^{r+2} - a_3(r+3)x^{r+3} + \sum_{n=4}^{+\infty}[(n+r)(n+r-1)a_n - (n+r)a_n + 4a_{n-4}]x^{n+r} = 0$$

$a_0x^r[r(r-1)-r]=0$

$r(r-1-1)=0 \quad \Rightarrow \quad r_1=0 \quad , \quad r_2=2 \Rightarrow \boldsymbol{a_0 \neq 0}$

$a_1x^{r+1}[r(r+1)-(r+1)]=0$

$a_1x^{r+1}[2(2+1)-(2+1)]=0 \Rightarrow a_1x^{r+1}(3)=0 \Rightarrow \boldsymbol{a_1=0}$

$a_2x^{r+2}[(r+1)(r+2)-(r+2)]=0$

$a_2x^{r+2}[(2+1)(2+2)-(2+2)]=0 \Rightarrow a_2x^{r+2}(8)=0 \Rightarrow \boldsymbol{a_2=0}$

$a_3x^{r+3}[(r+3)(r+2)-(r+3)]=0$

$a_3x^{r+3}[(2+3)(2+2)-(2+3)]=0 \Rightarrow a_2x^{r+2}(15)=0 \Rightarrow \boldsymbol{a_3=0}$

$[(n+r)(n+r-1)a_n-(n+r)a_n+4a_{n-4}]x^{n+r}=0$

$(n+r)(n+r-1)a_n-(n+r)a_n+4a_{n-4}=0 \quad \Rightarrow \quad \boldsymbol{a_n(r)=\frac{4a_{n-4}}{(n+r)(2-n-r)}} \; ; \forall \boldsymbol{n} \geq \boldsymbol{4}$

<u>$Para \;\; r=2$</u>

$a_n=-\frac{4a_{n-4}}{(n+2)(n)} \; ; \forall n \geq 4$

$n=4 \quad \Rightarrow \quad a_4=-\frac{4a_0}{4*6}$

$n=5 \quad \Rightarrow \quad a_5=-\frac{4a_1}{5*7} \quad \Rightarrow \quad a_5=0$

$n=6 \quad \Rightarrow \quad a_6=-\frac{4a_2}{6*8} \quad \Rightarrow \quad a_6=0$

$n=7 \quad \Rightarrow \quad a_7=-\frac{4a_3}{7*9} \quad \Rightarrow \quad a_7=0$

$n=8 \quad \Rightarrow \quad a_8=-\frac{4a_4}{8*10} \quad \Rightarrow \quad a_8=\frac{4*4a_0}{4*6*8*10}$

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$n=12 \quad \Rightarrow \quad a_{12}=-\frac{4a_8}{12*14} \quad \Rightarrow \quad a_{12}=\frac{4*4*4a_0}{4*6*8*10*12*14}$

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Entonces:

$$y_1(x)=\sum_{n=0}^{+\infty} a_n\, x^{n+r}$$

$$y_1(x)=\sum_{n=0}^{+\infty} a_n\, x^{n+2}$$

$$y_1(x)=a_0x^2+a_1x^3+a_2x^4+a_3x^5+a_4x^6+a_5x^7+\ \dots\dots\dots\dots\dots\dots\dots\dots$$

$$y_1(x)=a_0x^2+a_1x^3+a_2x^4+a_3x^5+a_4x^6+a_5x^7+\ \dots\dots\dots\dots\dots\dots\dots\dots$$

$$y_1(x)=a_0x^2-\frac{4a_0}{4*6}x^6+\frac{4*4a_0}{4*6*8*10}x^{10}-\frac{4*4*4a_0}{4*6*8*10*12*14}x^{14}+\ \dots\dots\dots\dots\dots\dots\dots\dots$$

$$y_1(x)=a_0\left(x^2-\frac{4}{3!*2^2}x^6+\frac{4^2}{5!*2^4}x^{10}-\frac{4^3}{7!*2^6}x^{14}+\ \dots\dots\dots\dots\dots\dots\dots\dots.\right)$$

$$y_1(x)=a_0\sum_{n=0}^{+\infty}\frac{x^{4n+2}*4^n*(-1)^n}{(2n+1)!*2^{2n}}$$

$$y_1(x)=a_0\sum_{n=0}^{+\infty}\frac{(-1)^n(x^2)^{2n+1}}{(2n+1)!}$$

$$y_1(x)=a_0\,Sen(x^2)$$

$$y_2(x)=v(x)y_1(x)$$

Encontrando v(x)

$$v(x)=\int\frac{e^{-\int p(x)dx}}{{y_1}^2}dx \quad\Rightarrow\quad v(x)=\int\frac{e^{-\int -\frac{1}{x}dx}}{Sen(x^2)^2}dx$$

$$v(x)=\int\frac{e^{\int\frac{1}{x}dx}}{Sen(x^2)^2}dx \quad\Rightarrow\quad v(x)=\int\frac{e^{ln|x|}}{Sen(x^2)^2}dx$$

$$v(x)=\int\frac{x}{Sen(x^2)^2}dx$$

Integrando por cambio de variable:

$$u=x^2 \qquad du=2x\,dx$$

$$v(x)=\int\frac{1}{2}\frac{du}{Sen(u)^2} \quad\Rightarrow\quad v(x)=-\frac{1}{2}Cot(u) \quad\Rightarrow\quad v(x)=-\frac{1}{2}Cot(x^2)$$

$$y_2(x)=-\frac{1}{2}Cot(x^2)\,Sen(x^2)=-\frac{1}{2}Cos(x^2)$$

$$\boxed{\boldsymbol{y_1(x)=Sen(x^2)}}$$

$$\boxed{\boldsymbol{y_2(x)=-\frac{1}{2}Cos(x^2)}}$$

2) $\boldsymbol{2xy'' + (1+x)y' + y = 0}$

$$\lim_{x\to 0} x\,\frac{q(x)}{p(x)} \quad \Rightarrow \quad \lim_{x\to 0} x\,\frac{(1+x)}{2x} = \frac{1}{2}$$

$$\lim_{x\to 0} x^2\,\frac{r(x)}{p(x)} \quad \Rightarrow \quad \lim_{x\to 0} x^2\,\frac{1}{2x} = 0$$

$$y(x) = \sum_{n=0}^{+\infty} a_n\, x^{n+r}$$

$$y'(x) = \sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r-2}$$

$$2x\sum_{n=0}^{+\infty}(n+r)(n+r-1)a_n\, x^{n+r-2} + (1+x)\sum_{n=0}^{+\infty}(n+r)a_n\, x^{n+r-1} + \sum_{n=0}^{+\infty} a_n\, x^{n+r} = 0$$

$$2\sum_{n=0}^{+\infty}(n+r)(n+r-1)a_n\, x^{n+r-1} + \sum_{n=0}^{+\infty}(n+r)a_n\, x^{n+r-1} + \sum_{n=0}^{+\infty}(n+r)a_n\, x^{n+r} + \sum_{n=0}^{+\infty} a_n\, x^{n+r} = 0$$

<u>Multiplicando por “x”:</u>

$$2x\sum_{n=0}^{+\infty}(n+r)(n+r-1)a_n\, x^{n+r-1} + x\sum_{n=0}^{+\infty}(n+r)a_n\, x^{n+r-1} + x\sum_{n=0}^{+\infty}(n+r)a_n\, x^{n+r} + x\sum_{n=0}^{+\infty} a_n\, x^{n+r} = 0$$

$$2\sum_{n=0}^{+\infty}(n+r)(n+r-1)a_n\, x^{n+r} + \sum_{n=0}^{+\infty}(n+r)a_n\, x^{n+r} + \sum_{n=0}^{+\infty}(n+r)a_n\, x^{n+r+1} + \sum_{n=0}^{+\infty} a_n\, x^{n+r+1} = 0$$

$$M = n+1$$

$$2\sum_{n=0}^{+\infty}(n+r)(n+r-1)a_n\, x^{n+r} + \sum_{n=0}^{+\infty}(n+r)a_n\, x^{n+r} + \sum_{M=1}^{+\infty}(M-1+r)a_n\, x^{M+r} + \sum_{M=1}^{+\infty} a_{M-1}\, x^{M+r} = 0$$

<u>Generando términos hasta n=1</u>

$$2a_0(r)(r-1)x^r + a_0(r)x^r + \sum_{n=1}^{+\infty}[2(n+r)(n+r-1)a_n + (n+r)a_n + a_{n-1}(n-1+r) + a_{n-1}]x^{n+r} = 0$$

$a_0 x^r[2r(r-1)+r] = 0 \quad \Rightarrow \quad r(2r-2+1) = 0 \quad \Rightarrow \quad r_1 = 0 \quad , \quad r_2 = {}^1\!/_2 \Rightarrow \boldsymbol{a_0 \neq 0}$

$[2(n+r)(n+r-1)a_n + (n+r)a_n + a_{n-1}(n-1+r) + a_{n-1}]x^{n+r} = 0$

$2(n+r)(n+r-1)a_n + (n+r)a_n + a_{n-1}(n-1+r) + a_{n-1} = 0$

$$\boldsymbol{a_n(r) = -\frac{a_{n-1}(n-1+r+1)}{(n+r)[2(n+r-1)+1]}} \; ; \; \forall \boldsymbol{n \geq 1}$$

$a_n(r) = -\frac{a_{n-1}}{(2n+2r-1)}$; $\forall n \geq 1$

$Para \quad r = {}^{1}/_{2}$

$a_n = -\frac{a_{n-1}}{2n}$; $\forall n \geq 1$

$n = 1 \quad \Rightarrow \quad a_1 = -\frac{a_0}{2}$

$n = 2 \quad \Rightarrow \quad a_2 = -\frac{a_1}{4} \quad \Rightarrow \quad a_2 = \frac{a_0}{2*4}$

$n = 3 \quad \Rightarrow \quad a_3 = -\frac{a_2}{6} \quad \Rightarrow \quad a_3 = -\frac{a_0}{2*4*6}$

$n = 4 \quad \Rightarrow \quad a_7 = -\frac{a_3}{8} \quad \Rightarrow \quad a_7 = \frac{a_0}{2*4*6*8}$

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Entonces:

$$y_1(x) = \sum_{n=0}^{+\infty} a_n\, x^{n+r}$$

$$y_1(x) = \sum_{n=0}^{+\infty} a_n\, x^{n+\frac{1}{2}}$$

$$y_1(x) = a_0 x^{1/2} + a_1 x^{3/2} + a_2 x^{5/2} + a_3 x^{7/2} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.$$

$$y_1(x) = a_0 x^{1/2} - \frac{a_0}{2} x^{3/2} + \frac{a_0}{2*4} x^{5/2} - \frac{a_0}{2*4*6} x^{7/2} + \frac{a_0}{2*4*6*8} x^{9/2} - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.$$

$$y_1(x) = a_0 \left(x^{1/2} - \frac{1}{2} x^{3/2} + \frac{1}{2! * 2^2} x^{5/2} - \frac{1}{3! * 2^3} x^{7/2} + \frac{1}{4! * 2^4} x^{9/2} - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \right)$$

$$y_1(x) = a_0 \sum_{n=0}^{+\infty} \frac{(-1)^n x^{\frac{2n+1}{2}}}{n! * 2^n}$$

$$y_1(x) = a_0 \sum_{n=0}^{+\infty} \frac{(-1)^n x^n * x^{\frac{1}{2}}}{n! * 2^n}$$

$$y_1(x) = a_0 \sqrt{x} \sum_{n=0}^{+\infty} \frac{\left(-\frac{x}{2}\right)^n}{n!}$$

$$y_1(x) = a_0\, \sqrt{x}\, e^{-x/2}$$

$$y_2(x) = v(x)y_1(x)$$

$$v(x) = \int \frac{e^{-\int p(x)dx}}{{y_1}^2}dx \quad \Rightarrow \quad v(x) = \int \frac{e^{-\int \frac{1+x}{2x}dx}}{\left(\sqrt{x}\, e^{-x/2}\right)^2}dx$$

$$v(x) = \int \frac{e^{-\frac{1}{2}(ln|x|+x)}}{xe^{-x}}dx \quad \Rightarrow \quad v(x) = \int \frac{x^{-1/2}\, e^{-x/2}}{xe^{-x}}dx$$

$$v(x) = \int x^{-3/2}\, e^{x/2}\ dx$$

$$Para\ resolver \int x^{-3/2}\, e^{x/2}\ dx\ \ es\ necesario\ utilizar\ series$$

$$e^{x/2} = \sum_{n=0}^{+\infty} \frac{\left(\frac{x}{2}\right)^n}{n!}$$

$$\frac{e^{x/2}}{x^{3/2}} = \sum_{n=0}^{+\infty} \frac{\left(\frac{1}{2}\right)^n x^{n-\frac{3}{2}}}{n!}$$

$$\int \frac{e^{x/2}}{x^{3/2}}dx = \int \left[\sum_{n=0}^{+\infty} \frac{\left(\frac{1}{2}\right)^n (x)^{\frac{2n-3}{2}}}{n!}\right]dx$$

$$\int \frac{e^{x/2}}{x^{3/2}}dx = \int \left[\frac{1}{x^{3/2}} + \frac{1}{2x^{1/2}} + \sum_{n=2}^{+\infty} \frac{\left(\frac{1}{2}\right)^n (x)^{\frac{2n-3}{2}}}{n!}\right]dx$$

$$\int \frac{e^{x/2}}{x^{3/2}}dx = -\frac{2}{x^{3/2}} + \frac{x^{1/2}}{2} + \sum_{n=2}^{+\infty} \frac{\left(\frac{1}{2}\right)^n (x)^{\frac{2n-1}{2}}}{\left(\frac{2n-1}{2}\right)n!}$$

$$v(x) = -\frac{2}{x^{3/2}} + \frac{x^{1/2}}{2} + \sum_{n=2}^{+\infty} \frac{\left(\frac{1}{2}\right)^n (x)^{\frac{2n-1}{2}}}{\left(\frac{2n-1}{2}\right)n!}$$

$$y_2(x) = \sqrt{x}\, e^{-x/2}\left[-\frac{2}{x^{3/2}} + \frac{x^{1/2}}{2} + \sum_{n=2}^{+\infty} \frac{\left(\frac{1}{2}\right)^n (x)^{\frac{2n-1}{2}}}{\left(\frac{2n-1}{2}\right)n!}\right]$$

$$\boxed{\boldsymbol{y_1(x) = \sqrt{x}\, e^{-x/2}}}$$

$$\boxed{\boldsymbol{y_2(x) = \sqrt{x}\, e^{-x/2}\left[-\frac{2}{x^{3/2}} + \frac{x^{1/2}}{2} + \sum_{n=2}^{+\infty} \frac{\left(\frac{1}{2}\right)^n (x)^{\frac{2n-1}{2}}}{\left(\frac{2n-1}{2}\right)n!}\right]}}$$

3) $\boldsymbol{xy'' + (3-x)y' - y = 0}$

$$\lim_{x\to 0} x\,\frac{q(x)}{p(x)} \;\Rightarrow\; \lim_{x\to 0} x\,\frac{(3-x)}{x} = 3$$

$$\lim_{x\to 0} x^2\,\frac{r(x)}{p(x)} \;\Rightarrow\; \lim_{x\to 0} x^2\,\frac{-1}{x} = 0$$

$$y(x) = \sum_{n=0}^{+\infty} a_n\, x^{n+r}$$

$$y'(x) = \sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r-2}$$

$$x\sum_{n=0}^{+\infty}(n+r)(n+r-1)a_n\, x^{n+r-2} + (3-x)\sum_{n=0}^{+\infty}(n+r)a_n\, x^{n+r-1} - \sum_{n=0}^{+\infty} a_n\, x^{n+r} = 0$$

$$\sum_{n=0}^{+\infty}(n+r)(n+r-1)a_n\, x^{n+r-1} + 3\sum_{n=0}^{+\infty}(n+r)a_n\, x^{n+r-1} - \sum_{n=0}^{+\infty}(n+r)a_n\, x^{n+r} - \sum_{n=0}^{+\infty} a_n\, x^{n+r} = 0$$

Multiplicado por "x"

$$x\sum_{n=0}^{+\infty}(n+r)(n+r-1)a_n\, x^{n+r-1} + 3x\sum_{n=0}^{+\infty}(n+r)a_n\, x^{n+r-1} - x\sum_{n=0}^{+\infty}(n+r)a_n\, x^{n+r} - x\sum_{n=0}^{+\infty} a_n\, x^{n+r} = 0$$

$$\sum_{n=0}^{+\infty}(n+r)(n+r-1)a_n\, x^{n+r} + 3\sum_{n=0}^{+\infty}(n+r)a_n\, x^{n+r} - \sum_{n=0}^{+\infty}(n+r)a_n\, x^{n+r+1} - \sum_{n=0}^{+\infty} a_n\, x^{n+r+1} = 0$$

$$M = n+1$$

$$\sum_{n=0}^{+\infty}(n+r)(n+r-1)a_n\, x^{n+r} + 3\sum_{n=0}^{+\infty}(n+r)a_n\, x^{n+r} - \sum_{M=1}^{+\infty}(M-1+r)a_n\, x^{M+r} - \sum_{M=1}^{+\infty} a_{M-1}\, x^{M+r} = 0$$

Generando términos hasta n=1

$$a_0(r)(r-1)x^r + 3a_0(r)x^r + \sum_{n=1}^{+\infty}[(n+r)(n+r-1)a_n + 3(n+r)a_n - a_{n-1}(n-1+r) - a_{n-1}]x^{n+r} = 0$$

$$a_0x^r[r(r-1)+3r] = 0 \;\Longrightarrow\; r(r-1+3) = 0 \;\Longrightarrow\; r_1 = 0 \;\;,\;\; r_2 = -2 \;\Longrightarrow \boldsymbol{a_0 \neq 0}$$

$$[(n+r)(n+r-1)a_n + 3(n+r)a_n - a_{n-1}(n-1+r) - a_{n-1}]x^{n+r} = 0$$

$$a_n(r) = \frac{a_{n-1}(n-1+r+1)}{(n+r)[(n+r-1)+3]} \;;\; \forall n \geq 1$$

$$\boldsymbol{a_n(r) = \frac{a_{n-1}}{(n+r+2)} \;;\; \forall n \geq 1}$$

$\underline{Para\ \ r = 0}$

$a_n = \dfrac{a_{n-1}}{n+2}$; $\forall n \geq 1$

$n = 1 \implies a_1 = \frac{a_0}{3}$

$n = 2 \implies a_2 = \frac{a_1}{4} \implies a_2 = \frac{a_0}{3*4}$

$n = 3 \implies a_3 = \frac{a_2}{5} \implies a_3 = \frac{a_0}{3*4*5}$

$n = 4 \implies a_7 = \frac{a_3}{6} \implies a_7 = \frac{a_0}{3*4*5*6}$

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Entonces:

$$y_1(x) = \sum_{n=0}^{+\infty} a_n\, x^{n+r}$$

$$y_1(x) = \sum_{n=0}^{+\infty} a_n\, x^{n}$$

$$y_1(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ \dots\dots\dots\dots\dots\dots\dots\dots$$

$$y_1(x) = a_0 + \frac{a_0}{3}\,x + \frac{a_0}{3*4}\,x^2 + \frac{a_0}{3*4*5}x^3 + \frac{a_0}{3*4*5*6}\,x^4 + \ \dots\dots\dots\dots\dots\dots\dots\dots$$

$$y_1(x) = a_0\left(1 + \frac{x}{3} + \frac{x^2}{3*4} + \frac{x^3}{3*4*5} + \frac{x^4}{3*4*5*6} + \ \dots\dots\dots\dots\dots\dots\dots\dots.\right)$$

$$y_1(x) = 2a_0\left(\frac{1}{2} + \frac{x}{2*3} + \frac{x^2}{2*3*4} + \frac{x^3}{2*3*4*5} + \frac{x^4}{2*3*4*5*6} + \ \dots\dots\dots\dots\dots\dots\dots\dots.\right)$$

$$y_1(x) = 2a_0 \sum_{n=0}^{+\infty} \frac{x^n}{(n+2)!}$$

Como no sabemos a que converge la sumatoria probemos con $r_2 = -2$, *siempre y cuando* $a_n(r) = \dfrac{a_{n-1}}{(n+r+2)}$ *exista*, $\forall n \geq 1$

$\underline{Para\ \ r = -2}$

$a_n = \dfrac{a_{n-1}}{n}$; $\forall n \geq 1$

$n = 1 \implies a_1 = a_0$

$n = 2 \implies a_2 = \frac{a_1}{4} \implies a_2 = \frac{a_0}{2}$

$n = 3 \implies a_3 = \frac{a_2}{5} \implies a_3 = \frac{a_0}{2*3}$

$n = 4 \implies a_7 = \frac{a_3}{6} \implies a_7 = \frac{a_0}{2*3*4}$

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$$y_2(x) = \sum_{n=0}^{+\infty} a_n\, x^{n+r}$$

$$y_2(x) = \sum_{n=0}^{+\infty} a_n\, x^{n-2}$$

$$y_2(x) = a_0 x^{-2} + a_1 x^{-1} + a_2 + a_3 x + a_4 x^2 + \ \dots\dots\dots\dots\dots\dots\dots\dots$$

$$y_2(x) = a_0 x^{-2} + a_0 x^{-1} + \frac{a_0}{2} + \frac{a_0}{2*3} x + \frac{a_0}{2*3*4}\, x^2 + \ \dots\dots\dots\dots\dots\dots\dots\dots$$

$$y_2(x) = a_0\left(x^{-2} + x^{-1} + \frac{1}{2} + \frac{x}{2*3} + \frac{x^2}{2*3*4} + \ \dots\dots\dots\dots\dots\dots\dots.\right)$$

$$y_2(x) = a_0 \sum_{n=0}^{+\infty} \frac{x^{n-2}}{n!} \quad \Rightarrow \quad y_2(x) = a_0 \sum_{n=0}^{+\infty} \frac{x^n * x^{-2}}{n!}$$

$$y_2(x) = a_0 \frac{1}{x^2} \sum_{n=0}^{+\infty} \frac{x^n}{n!} \quad \Rightarrow \quad y_2(x) = a_0 \frac{e^x}{x^2}$$

$$y_1(x) = v(x) y_2(x)$$

$$v(x) = \int \frac{e^{-\int p(x)dx}}{{y_2}^2} dx \quad \Rightarrow \quad v(x) = \int \frac{e^{-\int \frac{3-x}{x}dx}}{\left(\frac{e^x}{x^2}\right)^2} dx$$

$$v(x) = \int \frac{e^{-\int \frac{3-x}{x}dx}}{\left(\frac{e^x}{x^2}\right)^2} dx \quad \Rightarrow \quad v(x) = \int \frac{e^{(-3\, ln|x| + x)}}{\frac{e^{2x}}{x^4}} dx$$

$$v(x) = \int \frac{x^4 x^{-3}\, e^x}{e^{2x}} dx \quad \Rightarrow \quad v(x) = \int \frac{x}{e^x} dx$$

<u>Integrando por partes:</u>

$$u = x \qquad du = dx$$

$$dv = \frac{dx}{e^x} \qquad v = -\frac{1}{e^x}$$

$$v(x) = -\frac{x}{e^x} - \int -\frac{dx}{e^x} \quad \Rightarrow \quad v(x) = -\frac{x}{e^x} - \frac{1}{e^x}$$

$$y_1(x) = -\left(\frac{x}{e^x} + \frac{1}{e^x}\right)\frac{e^x}{x^2}$$

$$\boxed{\boldsymbol{y_1(x) = -\left(\frac{1}{x} + \frac{1}{x^2}\right)}}$$

$$\boxed{\boldsymbol{y_2(x) = \frac{e^x}{x^2}}}$$

4) $\boldsymbol{x(1-x)y'' - 3y' + 2y = 0}$

$$\lim_{x\to 0} x\,\frac{q(x)}{p(x)} \;\Rightarrow\; \lim_{x\to 0} x\,\frac{-3}{x(1-x)} = -3$$

$$\lim_{x\to 0} x^2\,\frac{r(x)}{p(x)} \;\Rightarrow\; \lim_{x\to 0} x^2\,\frac{2}{x(1-x)} = 0$$

$$y(x) = \sum_{n=0}^{+\infty} a_n\,x^{n+r}$$

$$y'(x) = \sum_{n=0}^{+\infty} (n+r)a_n\,x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\,x^{n+r-2}$$

$$x(1-x)\sum_{n=0}^{+\infty}(n+r)(n+r-1)a_n\,x^{n+r-2} - 3\sum_{n=0}^{+\infty}(n+r)a_n\,x^{n+r-1} + 2\sum_{n=0}^{+\infty}a_n\,x^{n+r} = 0$$

$$x\sum_{n=0}^{+\infty}(n+r)(n+r-1)a_n\,x^{n+r-2} - x^2\sum_{n=0}^{+\infty}(n+r)(n+r-1)a_n\,x^{n+r-2} - 3\sum_{n=0}^{+\infty}(n+r)a_n\,x^{n+r-1} + 2\sum_{n=0}^{+\infty}a_n\,x^{n+r} = 0$$

$$\sum_{n=0}^{+\infty}(n+r)(n+r-1)a_n\,x^{n+r-1} - \sum_{n=0}^{+\infty}(n+r)(n+r-1)a_n\,x^{n+r} - 3\sum_{n=0}^{+\infty}(n+r)a_n\,x^{n+r-1} + 2\sum_{n=0}^{+\infty}a_n\,x^{n+r} = 0$$

<u>Multiplicando por "x"</u>

$$x\sum_{n=0}^{+\infty}(n+r)(n+r-1)a_n\,x^{n+r-1} - x\sum_{n=0}^{+\infty}(n+r)(n+r-1)a_n\,x^{n+r} - 3x\sum_{n=0}^{+\infty}(n+r)a_n\,x^{n+r-1} + 2x\sum_{n=0}^{+\infty}a_n\,x^{n+r} = 0$$

$$\sum_{n=0}^{+\infty}(n+r)(n+r-1)a_n\,x^{n+r} - \sum_{n=0}^{+\infty}(n+r)(n+r-1)a_n\,x^{n+r+1} - 3\sum_{n=0}^{+\infty}(n+r)a_n\,x^{n+r} + 2\sum_{n=0}^{+\infty}a_n\,x^{n+r+1} = 0$$

$$M = n+1$$

$$\sum_{n=0}^{+\infty}(n+r)(n+r-1)a_n\,x^{n+r} - \sum_{M=1}^{+\infty}(M-1+r)(M+r-2)a_{M-1}\,x^{n+r} - 3\sum_{n=0}^{+\infty}(n+r)a_n\,x^{n+r} + 2\sum_{M=1}^{+\infty}a_{M-1}\,x^{M+r} = 0$$

$$a_0(r)(r-1)x^r - 3a_0(r)x^r + \sum_{n=1}^{+\infty}[(n+r)(n+r-1)a_n - (n-1+r)(n+r-2)a_{n-1} - 3a_n(n+r) + 2a_{n-1}]x^{n+r} = 0$$

$$a_0x^r[r(r-1)-3r] = 0 \;\Longrightarrow\; r(r-1-3) = 0 \;\Longrightarrow\; r_1 = 0 \;\;,\;\; r_2 = 4 \Longrightarrow \boldsymbol{a_0 \neq 0}$$

$$[(n+r)(n+r-1)a_n - (n-1+r)(n+r-2)a_{n-1} - 3a_n(n+r) + 2a_{n-1}]x^{n+r} = 0$$

$$a_n(r) = \frac{a_{n-1}[(n-1+r)(n+r-2)-2]}{(n+r)[(n+r-1)-3]}\;;\;\forall n \geq 1$$

$$a_n(r) = \frac{a_{n-1}[(n+r)^2 - 3(n+r)]}{(n+r)(n+r-4)}\;;\;\forall n \geq 1 \quad\Rightarrow\quad \boldsymbol{a_n(r) = \frac{a_{n-1}(n+r-3)}{(n+r-4)}\;;\;\forall n \geq 1}$$

$\underline{Para \;\; r = 4}$

$a_n = \dfrac{a_{n-1}(n+1)}{n}$; $\forall n \geq 1$

$n = 1 \quad \Longrightarrow \quad a_1 = 2a_0$

$n = 2 \quad \Longrightarrow \quad a_2 = \frac{3a_1}{2} \quad \Longrightarrow \quad a_2 = 3a_0$

$n = 3 \quad \Longrightarrow \quad a_3 = \frac{4a_2}{3} \quad \Longrightarrow \quad a_3 = 4a_0$

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$$y_1(x) = \sum_{n=0}^{+\infty} a_n \, x^{n+r}$$

$$y_1(x) = \sum_{n=0}^{+\infty} a_n \, x^{n+4}$$

$y_1(x) = a_0x^4 + a_1x^5 + a_2x^6 + a_3x^7 + a_4x^8 + \; \ldots \ldots \ldots \ldots \ldots \ldots \ldots .$

$y_1(x) = a_0x^4 + a_1x^5 + a_2x^6 + a_3x^7 + a_4x^8 + \; \ldots \ldots \ldots \ldots \ldots \ldots \ldots .$

$y_1(x) = a_0x^4 + 2a_0x^5 + 3a_0x^6 + 4a_0x^7 + 5a_0x^8 + \; \ldots \ldots \ldots \ldots \ldots \ldots \ldots .$

$y_1(x) = a_0(x^4 + 2x^5 + 3x^6 + 4x^7 + 5x^8 + \; \ldots \ldots \ldots \ldots \ldots \ldots \ldots .$

$Sabemos \; que$:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \; \ldots \ldots \ldots .$$

Derivando tenemos:

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}(1 + x + x^2 + x^3 + x^4 + x^5 + \; \ldots \ldots \ldots \ldots .)$$

$$-\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \; \ldots \ldots \ldots .$$

$$-\frac{x^4}{(1-x)^2} = x^4 + 2x^5 + 3x^6 + 4x^7 + 5x^8 + \; \ldots \ldots \ldots .$$

$$y_1(x) = -a_0\left[\frac{x^4}{(1-x)^2}\right]$$

$$y_2(x) = v(x)y_1(x)$$

$$v(x)=\int \frac{e^{-\int p(x)dx}}{{y_1}^2}dx \quad \Rightarrow \quad v(x)=\int \frac{e^{-\int -\frac{3}{x(x-1)}dx}}{\left[\frac{x^4}{(1-x)^2}\right]^2}dx$$

$$v(x)=\int \frac{e^{\int\left(\frac{3}{x}+\frac{3}{1-x}\right)dx}}{\frac{x^8}{(1-x)^4}}dx \quad \Rightarrow \quad v(x)=\int \frac{e^{(3\,ln|x|-3\,ln|1-x|)}}{\frac{x^8}{(1-x)^4}}dx$$

$$v(x)=\int \frac{(1-x)^4x^3(1-x)^{-3}}{x^8}dx \quad \Rightarrow \quad v(x)=\int \frac{(1-x)}{x^5}dx$$

$$v(x)=-\frac{1}{4x^4}+\frac{1}{3x^3}$$

$$y_2(x)=\left(\frac{1}{3x^3}-\frac{1}{4x^4}\right)\frac{x^4}{(1-x)^2}$$

$$\boxed{\boldsymbol{y_1(x)=\frac{x^4}{(1-x)^2}}}$$

$$\boxed{\boldsymbol{y_2(x)=\left[\frac{x}{3(1-x)^2}-\frac{1}{4(1-x)^2}\right]}}$$

5) $\boldsymbol{xy'' + 2y' - xy = 0}$

$$\lim_{x\to 0} x\,\frac{q(x)}{p(x)} \;\Rightarrow\; \lim_{x\to 0} x\,\frac{2}{x} = 2$$

$$\lim_{x\to 0} x^2\,\frac{r(x)}{p(x)} \;\Rightarrow\; \lim_{x\to 0} x^2\,\frac{-x}{x} = 0$$

$$y(x) = \sum_{n=0}^{+\infty} a_n\, x^{n+r}$$

$$y'(x) = \sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r-2}$$

$$x\sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r-2} + 2\sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r-1} - x\sum_{n=0}^{+\infty} a_n\, x^{n+r} = 0$$

$$\sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r-1} + 2\sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r-1} - \sum_{n=0}^{+\infty} a_n\, x^{n+r+1} = 0$$

Multiplicando por "x":

$$x\sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r-1} + 2x\sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r-1} - x\sum_{n=0}^{+\infty} a_n\, x^{n+r+1} = 0$$

$$\sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r} + 2\sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r} - \sum_{n=0}^{+\infty} a_n\, x^{n+r+2} = 0$$

$$M = n+2$$

$$\sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r} + 2\sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r} - \sum_{M=2}^{+\infty} a_{M-2}\, x^{M+2} = 0$$

$$a_0(r)(r-1)x^r + a_1(r+1)(r)x^{r+1} + 2a_0(r)x^r + 2a_1(r+1)x^{r+1}\sum_{n=2}^{+\infty}[(n+r)(n+r-1)a_n + 2a_n(n+r) - a_{n-2}]x^{n+r} = 0$$

$$a_0x^r[r(r-1)+r] = 0$$

$$r(r-1+1) = 0 \;\;\Longrightarrow\;\; r_1 = 0 \;\;,\;\; r_2 = 0 \Longrightarrow \boldsymbol{a_0 \neq 0}$$

$$a_1x^{r+1}[r(r+1) + 2(r+1)] = 0$$

$$a_1x^{r+1}[0(0+1) + 2(0+1)] = 0 \Longrightarrow a_1x^{r+1}(2) = 0 \Longrightarrow \boldsymbol{a_1 = 0}$$

$[(n+r)(n+r-1)a_n + 2a_n(n+r) - a_{n-2}]x^{n+r} = 0$

$(n+r)(n+r-1)a_n + 2a_n(n+r) - a_{n-2} = 0$

$a_n(r) = \dfrac{a_{n-2}}{(n+r)[(n+r-1)+2]}$; $\forall n \geq 2$

$a_n(r) = \dfrac{a_{n-2}}{(n+r)(n+r+1)}$; $\forall n \geq 2$

$\boldsymbol{a_n(r) = \dfrac{a_{n-2}}{(n+r)(n+r+1)}}$; $\boldsymbol{\forall n \geq 2}$

$\underline{Para\ \ r = 0}$

$a_n = \dfrac{a_{n-2}}{n(n+1)}$; $\forall n \geq 2$

$n = 2 \quad \Longrightarrow \quad a_2 = \frac{a_0}{2*3}$

$n = 3 \quad \Longrightarrow \quad a_3 = \frac{a_1}{2*3} \quad \Longrightarrow \quad a_3 = 0$

$n = 4 \quad \Longrightarrow \quad a_4 = \frac{a_2}{4*5} \quad \Longrightarrow \quad a_4 = \frac{a_0}{2*3*4*5}$

$n = 5 \quad \Longrightarrow \quad a_5 = \frac{a_3}{5*6} \quad \Longrightarrow \quad a_5 = 0$

$n = 6 \quad \Longrightarrow \quad a_6 = \frac{a_4}{6*7} \quad \Longrightarrow \quad a_6 = \frac{a_0}{2*3*4*5*6*7}$

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$y_1(x) = \sum_{n=0}^{+\infty} a_n\, x^{n+r}$

$y_1(x) = \sum_{n=0}^{+\infty} a_n\, x^{n}$

$y_1(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ a_5x^5 + a_66 + \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$y_1(x) = a_0 + \dfrac{a_0}{2*3}\ \ x^2\ + \dfrac{a_0}{2*3*4*5}x^4 + \dfrac{a_0}{2*3*4*5*6*7}x^6 + \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$y_1(x) = a_0\left(\ 1 + \dfrac{x^2}{3!} + \dfrac{x^4}{5!} + \dfrac{x^6}{7!} + \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\right)$

$y_1(x) = a_0 \sum_{n=0}^{+\infty} \dfrac{x^{2n}}{(2n+1)!}$

$y_1(x) = a_0 \dfrac{1}{x} \sum_{n=0}^{+\infty} \dfrac{x^{2n+1}}{(2n+1)!}$

$y_1(x) = a_0 \dfrac{Senh\ x}{x}$

$y_2(x) = v(x)y_1(x)$

$v(x) = \int \frac{e^{-\int p(x)dx}}{{y_1}^2}dx \quad \Rightarrow \quad v(x) = \int \frac{e^{-\int \frac{2}{x}dx}}{\left(\frac{Senh\ x}{x}\right)^2}dx$

$v(x) = \int \frac{e^{-2\ ln|x|}x^2}{(Senh\ x)^2}dx \quad \Rightarrow \quad v(x) = \int \frac{x^2x^{-2}}{(Senh\ x)^2}dx$

$v(x) = \int (Senh\ x)^{-2}\ dx$

$Pero\ Senh\ x = \frac{e^x - e^{-x}}{2}\ \ , entonces:$

$v(x) = \int \left(\frac{2}{e^x - e^{-x}}\right)^2 dx \quad \Rightarrow \quad v(x) = \int \left(\frac{2e^x}{e^{2x} - 1}\right)^2 dx \quad \Rightarrow \quad v(x) = \int \frac{4e^{2x}}{(e^{2x} - 1)^2}dx$

Integrando por cambio de variable:

$u = e^x \quad du = e^x$

$v(x) = \int \frac{4u^2}{(u^2 - 1)^2}dx \quad \Rightarrow \quad v(x) = \int \frac{4u^2}{[(u-1)(u+1)]^2}dx \quad \Rightarrow \quad v(x) = \int \frac{4u^2}{(u-1)^2(u+1)^2}dx$

Integrando aplicando fracciones parciales:

$\frac{u^2}{(u-1)^2(u+1)^2} = \frac{2A(u-1) + B}{(u-1)^2} + \frac{2C(u+1) + D}{(u+1)^2}$

$u^2 = [2A(u-1) + B](u+1)^2 + [2C(u+1) + D](u-1)^2$

$u^2 = 2A(u^3 + u^2 - u - 1) + B(u^2 + 2u + 1) + 2C(u^3 - u^2 - u + 1) + D(u^2 - 2u + 1)$

$u^2 = (2A + 2C)u^3 + (2A + B - 2C + D)u^2 + (-2A + 2B - 2C - 2D)u + (-2A + B + 2C + D)$

$(1)\ 2A + 2C = 0$

$(2)\ 2A + B - 2C + D = 1$

$(3) - 2A + 2B - 2C - 2D = 0$

$(4) - 2A + B + 2C + D = 0$

$2A = -2C \qquad (1) + (3)\quad 2B = 2D \qquad (2) + (3) = 2B + 2D = 1$

$B = \frac{1}{4} \qquad 2D + 2D = 1$

$D = \frac{1}{4}$

$2C + \frac{1}{4} + 2C + \frac{1}{4} = 0$

$C = \frac{1}{8} \qquad A = \frac{1}{8}$

Entonces:

$$v(x) = \int \left[\frac{2A(u-1)}{(u-1)^2} + \frac{B}{(u-1)^2} + \frac{2C(u+1)}{(u+1)^2} + \frac{D}{(u+1)^2}\right] dx$$

$$v(x) = 2A\ ln|(u-1)^2| - \frac{B}{(u-1)} + 2C\ ln|(u+1)^2| - \frac{D}{(u+1)}$$

$$v(x) = 2\left(\frac{1}{8}\right)\ ln|(e^x-1)^2| - \frac{1}{4(e^x-1)} + 2\left(\frac{1}{8}\right)\ ln|(e^x+1)^2| - \frac{1}{4(e^x+1)}$$

$$y_2(x) = \left[\frac{1}{4}\ ln|(e^x-1)^2| - \frac{1}{4(e^x-1)} + \frac{1}{4}\ ln|(e^x+1)^2| - \frac{1}{4(e^x+1)}\right]\frac{Senh\ x}{x}$$

$$\boxed{\boldsymbol{y_1(x) = \frac{Senh\ x}{x}}}$$

$$\boxed{\boldsymbol{y_2(x) = \frac{1}{4}\left[\ ln\left|(e^x-1)^2\right| - \frac{1}{(e^x-1)} + \ ln\left|(e^x+1)^2\right| - \frac{1}{(e^x+1)}\right]\frac{Senh\ x}{x}}}$$

6) $\boldsymbol{xy'' + 3y' + 4x^3y = 0}$

$$\lim_{x\to 0} x\,\frac{q(x)}{p(x)} \;\Rightarrow\; \lim_{x\to 0} x\,\frac{3}{x} = 3$$

$$\lim_{x\to 0} x^2\,\frac{r(x)}{p(x)} \;\Rightarrow\; \lim_{x\to 0} x^2\,\frac{4x^3}{x} = 0$$

$$y(x) = \sum_{n=0}^{+\infty} a_n\, x^{n+r}$$

$$y'(x) = \sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r-2}$$

$$x\sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r-2} + 3\sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r-1} + 4x^3\sum_{n=0}^{+\infty} a_n\, x^{n+r} = 0$$

$$\sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r-1} + 3\sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r-1} + 4\sum_{n=0}^{+\infty} a_n\, x^{n+r+3} = 0$$

<u>Multiplicando por "x"</u>

$$x\sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r-1} + 3x\sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r-1} + 4x\sum_{n=0}^{+\infty} a_n\, x^{n+r+3} = 0$$

$$\sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r} + 3\sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r} + 4\sum_{n=0}^{+\infty} a_n\, x^{n+r+4} = 0$$

$$M = n+4$$

$$\sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r} + 3\sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r} + 4\sum_{M=4}^{+\infty} a_{M-4}\, x^{M+r} = 0$$

<u>Generando términos hasta n=4</u>

$a_0(r)(r-1)x^r + a_1(r+1)(r)x^{r+1} + a_2(r+2)(R+1)x^{r+2} + a_3(r+3)(r+2)x^{r+3} +$

$3a_0(r)x^r + 3a_1(r+1)x^{r+1} + 3a_2(r+2)x^{r+2} + 3a_3(r+3)x^{r+3} + \sum_{n=4}^{+\infty}[(n+r)(n+r-1)a_n + 3(n+r)a_n + 4a_{n-4}]x^{n+r} = 0$

$a_0x^r[r(r-1)+3r] = 0$

$r(r-1+3) = 0 \;\Rightarrow\; r_1 = 0 \;,\; r_2 = -2 \;\Rightarrow \boldsymbol{a_0 \neq 0}$

$a_1x^{r+1}[r(r+1)+3(r+1)]=0$

$a_1x^{r+1}[0(0+1)+3(0+1)]=0 \Rightarrow a_1x^{r+1}(3)=0 \Rightarrow \boldsymbol{a_1=0}$

$a_2x^{r+2}[(r+1)(r+2)+3(r+2)]=0$

$a_2x^{r+2}[(0+1)(0+2)+3(0+2)]=0 \Rightarrow a_2x^{r+2}(6)=0 \Rightarrow \boldsymbol{a_2=0}$

$a_3x^{r+3}[(r+3)(r+2)+3(r+3)]=0$

$a_3x^{r+3}[(0+3)(0+2)+3(0+3)]=0 \Rightarrow a_2x^{r+2}(9)=0 \Rightarrow \boldsymbol{a_3=0}$

$[(n+r)(n+r-1)a_n+3(n+r)a_n+4a_{n-4}]x^{n+r}=0$

$(n+r)(n+r-1)a_n+3(n+r)a_n+4a_{n-4}=0$

$a_n(r)=-\frac{4a_{n-4}}{(n+r)(n+r-1+3)}\ ;\ \forall n\geq 4$

$\boldsymbol{a_n(r)=-\frac{4a_{n-4}}{(n+r)(n+r+2)}\ ;\ \forall n\geq 4}$

$\underline{Para\ \ r=0}$

$a_n=-\frac{4a_{n-4}}{(n+2)(n)}\ ;\ \forall n\geq 4$

$n=4 \quad \Rightarrow \quad a_4=-\frac{4a_0}{4*6}$

$n=5 \quad \Rightarrow \quad a_5=-\frac{4a_1}{5*7} \quad \Rightarrow \quad a_5=0$

$n=6 \quad \Rightarrow \quad a_6=-\frac{4a_2}{6*8} \quad \Rightarrow \quad a_6=0$

$n=7 \quad \Rightarrow \quad a_7=-\frac{4a_3}{7*9} \quad \Rightarrow \quad a_7=0$

$n=8 \quad \Rightarrow \quad a_8=-\frac{4a_4}{8*10} \quad \Rightarrow \quad a_8=\frac{4*4a_0}{4*6*8*10}$

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$n=12 \quad \Rightarrow \quad a_{12}=-\frac{4a_8}{12*14} \quad \Rightarrow \quad a_{12}=\frac{4*4*4a_0}{4*6*8*10*12*14}$

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$$y_1(x) = \sum_{n=0}^{+\infty} a_n\, x^{n+r}$$
$$y_1(x) = \sum_{n=0}^{+\infty} a_n\, x^{n}$$

$$y_1(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$y_1(x) = a_0 - \frac{4a_0}{4*6}x^4 + \frac{4*4a_0}{4*6*8*10}x^8 - \frac{4*4*4a_0}{4*6*8*10*12*14}x^{12} + \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$y_1(x) = a_0\left(1 - \frac{4}{3!*2^2}x^4 + \frac{4^2}{5!*2^4}x^8 - \frac{4^3}{7!*2^6}x^{12} + \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\right)$$

$$y_1(x) = a_0 \sum_{n=0}^{+\infty} \frac{x^{4n}*4^n*(-1)^n}{(2n+1)!*2^{2n}}$$

$$y_1(x) = a_0 \frac{1}{x^2} \sum_{n=0}^{+\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$$

$$y_1(x) = a_0 \frac{1}{x^2} \sum_{n=0}^{+\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}$$

$$y_1(x) = a_0\ \frac{Sen(x^2)}{x^2}$$

$$y_2(x) = v(x)y_1(x)$$

$$v(x) = \int \frac{e^{-\int p(x)dx}}{{y_1}^2}dx \quad \Rightarrow \quad v(x) = \int \frac{e^{-\int \frac{3}{x}dx}}{\left[\frac{Sen(x^2)}{x^2}\right]^2}dx$$

$$v(x) = \int \frac{e^{-3\,ln|x|}}{\left[\frac{Sen(x^2)}{x^2}\right]^2}dx \quad \Rightarrow \quad v(x) = \int \frac{x^{-3}x^4}{Sen(x^2)^2}dx \quad \Rightarrow \quad v(x) = \int x\, Csc^2(x^2)$$

Integrando por cambio de variable:

$$u = x^2 \qquad du = 2x\,dx$$

$$v(x) = \int \frac{1}{2}Csc^2(u)du \quad \Rightarrow \quad v(x) = -\frac{1}{2}Cot(u) \quad \Rightarrow \quad v(x) = -\frac{1}{2}Cot(x^2)$$

$$y_2(x) = -\frac{1}{2}Cot(x^2)\ \frac{Sen(x^2)}{x^2} = -\frac{Cos(x^2)}{2x^2}$$

$$\boxed{\boldsymbol{y_1(x) = \frac{Sen(x^2)}{x^2}}}$$

$$\boxed{\boldsymbol{y_2(x) = \frac{Cos(x^2)}{2x^2}}}$$

7) $xy'' + 2y' + xy = 0$

$$\lim_{x\to 0} x\,\frac{q(x)}{p(x)} \quad \Rightarrow \quad \lim_{x\to 0} x\,\frac{2}{x} = 2$$

$$\lim_{x\to 0} x^2\,\frac{r(x)}{p(x)} \quad \Rightarrow \quad \lim_{x\to 0} x^2\,\frac{x}{x} = 0$$

$$y(x) = \sum_{n=0}^{+\infty} a_n\, x^{n+r}$$

$$y'(x) = \sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r-2}$$

$$x\sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r-2} + 2\sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r-1} + x\sum_{n=0}^{+\infty} a_n\, x^{n+r} = 0$$

$$\sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r-1} + 2\sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r-1} + \sum_{n=0}^{+\infty} a_n\, x^{n+r+1} = 0$$

Multiplicando por "x"

$$x\sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r-1} + 2x\sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r-1} + x\sum_{n=0}^{+\infty} a_n\, x^{n+r+1} = 0$$

$$\sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r} + 2\sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r} + \sum_{n=0}^{+\infty} a_n\, x^{n+r+2} = 0$$

$$M = n+2$$

$$\sum_{n=0}^{+\infty} (n+r)(n+r-1)a_n\, x^{n+r} + 2\sum_{n=0}^{+\infty} (n+r)a_n\, x^{n+r} + \sum_{M=2}^{+\infty} a_{M-2}\, x^{M+2} = 0$$

Generando términos hasta n=2

$$a_0(r)(r-1)x^r + a_1(r+1)(r)x^{r+1} + 2a_0(r)x^r + 2a_1(r+1)x^{r+1}\sum_{n=2}^{+\infty}[(n+r)(n+r-1)a_n + 2a_n(n+r) + a_{n-2}]x^{n+r} = 0$$

$$a_0x^r[r(r-1)+r] = 0$$

$$r(r-1+1) = 0 \quad \Rightarrow \quad r_1 = 0 \quad , \quad r_2 = 0 \Rightarrow \boldsymbol{a_0 \neq 0}$$

$$a_1x^{r+1}[r(r+1)+2(r+1)] = 0$$

$$a_1x^{r+1}[0(0+1)+2(0+1)] = 0 \Rightarrow a_1x^{r+1}(2) = 0 \Rightarrow \boldsymbol{a_1 = 0}$$

$$[(n+r)(n+r-1)a_n + 2a_n(n+r) + a_{n-2}]x^{n+r} = 0$$

$$a_n(r) = -\frac{a_{n-2}}{(n+r)[(n+r-1)+2]} \ ; \ \forall n \geq 2$$

$$a_n(r) = -\frac{a_{n-2}}{(n+r)(n+r+1)} \ ; \ \forall n \geq 2$$

$$\boldsymbol{a_n(r) = -\frac{a_{n-2}}{(n+r)(n+r+1)} \ ; \ \forall n \geq 2}$$

$\underline{Para \ \ r = 0}$

$$a_n = -\frac{a_{n-2}}{n(n+1)} \ ; \ \forall n \geq 2$$

$$n = 2 \quad \Longrightarrow \quad a_2 = -\frac{a_0}{2*3}$$

$$n = 3 \quad \Longrightarrow \quad a_3 = -\frac{a_1}{2*3} \quad \Longrightarrow \quad a_3 = 0$$

$$n = 4 \quad \Longrightarrow \quad a_4 = -\frac{a_2}{4*5} \quad \Longrightarrow \quad a_4 = \frac{a_0}{2*3*4*5}$$

$$n = 5 \quad \Longrightarrow \quad a_5 = -\frac{a_3}{5*6} \quad \Longrightarrow \quad a_5 = 0$$

$$n = 6 \quad \Longrightarrow \quad a_6 = -\frac{a_4}{6*7} \quad \Longrightarrow \quad a_6 = \frac{a_0}{2*3*4*5*6*7}$$

.
.
.

$$y_1(x) = \sum_{n=0}^{+\infty} a_n \, x^{n+r}$$

$$y_1(x) = \sum_{n=0}^{+\infty} a_n \, x^{n}$$

$$y_1(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ a_5x^5 + a_66 + \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$y_1(x) = a_0 - \frac{a_0}{2*3} \ x^2 \ + \frac{a_0}{2*3*4*5}x^4 - \frac{a_0}{2*3*4*5*6*7}x^6 + \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$y_1(x) = a_0\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \right)$$

$$y_1(x) = a_0 \sum_{n=0}^{+\infty} \frac{x^{2n}(-1)^n}{(2n+1)!}$$

$$y_1(x) = a_0 \frac{1}{x} \sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$y_1(x) = a_0 \frac{Sen \ x}{x}$$

$y_2(x) = v(x)y_1(x)$

$$v(x) = \int \frac{e^{-\int p(x)dx}}{{y_1}^2}dx \quad \Rightarrow \quad v(x) = \int \frac{e^{-\int \frac{2}{x}dx}}{\left(\frac{Sen\,x}{x}\right)^2}dx$$

$$v(x) = \int \frac{e^{-2\,ln|x|}x^2}{(Sen\,x)^2}dx \quad \Rightarrow \quad v(x) = \int \frac{x^2x^{-2}}{(Sen\,x)^2}dx$$

$$v(x) = \int Csc^2(x)\,dx \quad \Rightarrow \quad v(x) = -Cot(x)$$

$$y_2(x) = -\frac{Sen\,x}{x}\,Cot(x) = -\frac{Cos\,(x)}{x}$$

$$\boxed{\boldsymbol{y_1(x) = \frac{Sen\,x}{x}}}$$

$$\boxed{\boldsymbol{y_2(x) = -\frac{Cos\,(x)}{x}}}$$

TRANSFORMADA DE LAPLACE

1) $\boldsymbol{\mathcal{L}\{Sen^5 t\}}$

Sabemos que:

① $e^{i\theta} = Cos\theta + i\, Sen\theta$

② $e^{-i\theta} = Cos\theta - i\, Sen\theta$

Entonces ① − ②

$$Sen\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\mathcal{L}\{Sen^5 t\} = \mathcal{L}\left\{\left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^5\right\}$$

$$\mathcal{L}\{Sen^5 t\} = \frac{1}{16}\mathcal{L}\left\{\frac{(e^{it})^5 - 5(e^{it})(e^{-it})^4 + 10(e^{it})^3(e^{-it})^2 - 10(e^{it})^2(e^{-it})^3 + 5(e^{it})(e^{-it})^4 - (e^{-it})^5}{2i}\right\}$$

$$\mathcal{L}\{Sen^5 t\} = \frac{1}{16}\mathcal{L}\left\{\frac{e^{5it} - 5e^{3it} + 10e^{it} - 10e^{-it} + 5e^{-3it} - e^{-5it}}{2i}\right\}$$

$$\mathcal{L}\{Sen^5 t\} = \frac{1}{16}\mathcal{L}\left\{\left(\frac{e^{5it} - e^{-5it}}{2i}\right) - 5\left(\frac{e^{3it} - e^{-3it}}{2i}\right) + 10\left(\frac{e^{it} - e^{-it}}{2i}\right)\right\}$$

$$\mathcal{L}\{Sen^5 t\} = \frac{1}{16}\mathcal{L}\{Sen(5t) - 5\, Sen(3t) + 10\, Sen(t)\}$$

$$\boxed{\boldsymbol{\mathcal{L}\{Sen^5 t\} = \frac{1}{16}\left(\frac{5}{s^2+25} - \frac{15}{s^2+9} + \frac{10}{s^2+1}\right)}}$$

2) $\boldsymbol{\mathcal{L}\{u(t-2\pi)Sen(t-2\pi)\}}$

Vamos a realizarlo paso a paso:

Como la función seno ya está desfasada, no hay problema, entonces, primero determinamos la trasformada de Laplace de la función seno: $\mathcal{L}\{Sen\ t\} = \frac{1}{s^2+1}$*, luego:*

$$\boxed{\boldsymbol{\mathcal{L}\{u(t-2\pi)Sen(t-2\pi)\} = e^{-2\pi s}\frac{1}{s^2+1}}}$$

$$3)\mathcal{L}\left\{u\left(t-\frac{\pi}{2}\right)e^{-2\left(t-\frac{\pi}{2}\right)}Cosh\,4\left(t-\frac{\pi}{2}\right)\right\}$$

Determinamos la transformada de Laplace del coseno hiperbólico

$$\mathcal{L}\{Cosh\,4t\}=\frac{s}{s^2-16}$$

Luego:

$$\mathcal{L}\{e^{-2t}Cosh\,4t\}=\frac{s+2}{(s+2)^2-16}$$

Y finalmente:

$$\boxed{\mathcal{L}\left\{u\left(t-\frac{\pi}{2}\right)e^{-2\left(t-\frac{\pi}{2}\right)}Cosh\,4\left(t-\frac{\pi}{2}\right)\right\}=e^{-\frac{\pi}{2}s}\left[\frac{s+2}{(s+2)^2-16}\right]}$$

$$4)\mathcal{L}\{u(t-2)t\}$$

Hay que desfasar la función

$$\mathcal{L}\{u(t-2)t\}=\mathcal{L}\{u(t-2)(t-2+2)\}$$

$$\mathcal{L}\{u(t-2)t\}=\mathcal{L}\{u(t-2)[(t-2)+2)]\}$$

$$\mathcal{L}\{u(t-2)t\}=\mathcal{L}\{u(t-2)(t-2)\}+2\mathcal{L}\{u(t-2)(1)\}$$

$$\boxed{\mathcal{L}\{u(t-2)t\}=e^{-2s}\frac{1}{s^2}+2e^{-2s}\frac{1}{s}}$$

$$5)\mathcal{L}\left\{u\left(t-\frac{\pi}{2}\right)Sen\,t\right\}$$

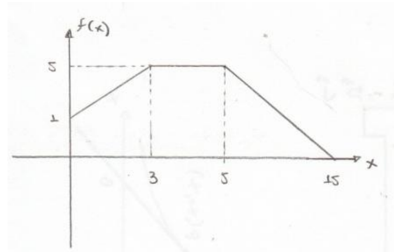
$$\mathcal{L}\left\{u\left(t-\frac{\pi}{2}\right)Sen\,t\right\}=\mathcal{L}\left\{u\left(t-\frac{\pi}{2}\right)Sen\left(\left(t-\frac{\pi}{2}\right)+\frac{\pi}{2}\right)\right\}$$

$$\mathcal{L}\left\{u\left(t-\frac{\pi}{2}\right)Sen\,t\right\}=\mathcal{L}\left\{u\left(t-\frac{\pi}{2}\right)\left[Sen\left(t-\frac{\pi}{2}\right)Cos\left(\frac{\pi}{2}\right)+Cos\left(t-\frac{\pi}{2}\right)Sen\left(\frac{\pi}{2}\right)\right]\right\}$$

$$\mathcal{L}\left\{u\left(t-\frac{\pi}{2}\right)Sen\,t\right\}=\mathcal{L}\left\{u\left(t-\frac{\pi}{2}\right)Cos\left(t-\frac{\pi}{2}\right)\right\}$$

$$\boxed{\mathcal{L}\left\{u\left(t-\frac{\pi}{2}\right)Sen\,t\right\}=e^{-\frac{\pi}{2}s}\frac{s}{s^2+1}}$$

6) $\boldsymbol{\mathcal{L}\{f(t)\}}$



$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{[u(t) - u(t-3)](t+2) + 5[u(t-3) - u(t-5)] + [u(t-5) - u(t-15)]\left(\frac{1}{2}t - \frac{15}{2}\right)\right\}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{u(t)(t+2) - u(t-3)(t+2) + 5u(t-3) - 5u(t-3) + u(t-5)\left(\frac{1}{2}t - \frac{15}{2}\right) - u(t-15)\left(\frac{1}{2}t - \frac{15}{2}\right)\right\}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{u(t)(t+2-2+2) - u(t-3)(t+2-5+5) + 5u(t-3) - 5u(t-5) + \frac{1}{2}u(t-5)(t-15+10-10) - \frac{1}{2}u(t-15)(t-15)\right\}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{u(t)t + 2u(t) - u(t-3)(t-3) - 5u(t-3) + 5u(t-3) - 5u(t-5) + \frac{1}{2}u(t-5)(t-5) - 10u(t-5) - \frac{1}{2}u(t-15)(t-15)\right\}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{u(t)t + 2u(t) - u(t-3)(t-3) - 15u(t-5) + \frac{1}{2}u(t-5)(t-5) - \frac{1}{2}u(t-15)(t-15)\right\}$$

$$\boxed{\boldsymbol{\mathcal{L}\{f(t)\} = \frac{1}{s^2} + \frac{2}{s} - e^{-3s}\left(\frac{1}{s^2}\right) - 15e^{-5s}\left(\frac{1}{s}\right) + \frac{e^{-5s}}{2}\left(\frac{1}{s^2}\right) - \frac{e^{-15s}}{2}\left(\frac{1}{s^2}\right)}}$$

7) $\boldsymbol{\mathcal{L}\left\{te^{-3t}Sen(4t)\right\}}$

$$\mathcal{L}\{Sen(4t)\} = \frac{4}{s^2 + 16}$$

$$\mathcal{L}\{e^{-3t}Sen(4t)\} = \frac{4}{(s+3)^2 + 16}$$

$$\mathcal{L}\{te^{-3t}Sen(4t)\} = -\frac{d}{ds}\left[\frac{4}{(s+3)^2 + 16}\right]$$

$$\mathcal{L}\{te^{-3t}Sen(4t)\} = -\frac{4}{[(s+3)^2 + 16]^2}[2(s+3)]$$

$$\boxed{\boldsymbol{\mathcal{L}\{te^{-3t}Sen(4t)\} = -\frac{8(s+3)}{[(s+3)^2 + 16]^2}}}$$

$$\mathbf{8)}\boldsymbol{\mathcal{L}}\left\{\boldsymbol{t}\int_{\mathbf{0}}^{\boldsymbol{t}}\boldsymbol{Sen(\tau)d\tau}\right\}$$

$$\mathcal{L}\left\{\int_0^t f(x)g(t-x)dx\right\}=F(s)G(s)$$

$$\mathcal{L}\left\{\int_0^t \overbrace{(1)}^{g(t-x)}\ \overbrace{Sen(\tau)}^{f(x)}\,d\tau\right\}=\frac{1}{s}\left(\frac{1}{s^2+1}\right)$$

$$\mathcal{L}\left\{t\int_0^t Sen(\tau)d\tau\right\}=-\frac{d}{ds}\left[\frac{1}{s}\left(\frac{1}{s^2+1}\right)\right]=\frac{1}{s^2}\left(\frac{1}{s^2+1}\right)+\frac{2s}{[s^2+1]^2}\left(\frac{1}{s}\right)$$

$$\boxed{\boldsymbol{\mathcal{L}}\left\{\boldsymbol{t}\int_{\mathbf{0}}^{\boldsymbol{t}}\boldsymbol{Sen(\tau)d\tau}\right\}=\frac{\mathbf{1}}{\boldsymbol{s^2}}\left(\frac{\mathbf{1}}{\boldsymbol{s^2}+\mathbf{1}}\right)+\frac{\mathbf{2}\boldsymbol{s}}{[\boldsymbol{s^2}+\mathbf{1}]^{2}}\left(\frac{\mathbf{1}}{\boldsymbol{s}}\right)}$$

$$\mathbf{9)}\boldsymbol{\mathcal{L}}\left\{\boldsymbol{e^{-2t}}\int_{\mathbf{0}}^{\boldsymbol{t}}\boldsymbol{\tau e^{-2\tau}Sen(\tau)d\tau}\right\}$$

$$\mathcal{L}\{Sen(t)\}=\frac{1}{s^2+1}$$

$$\mathcal{L}\{e^{-2t}Sen(t)\}=\frac{1}{(s+2)^2+1}$$

$$\mathcal{L}\{te^{-2t}Sen(t)\}=-\frac{d}{ds}\left[\frac{1}{(s+2)^2+1}\right]=\frac{2(s+2)}{[(s+2)^2+1]^2}$$

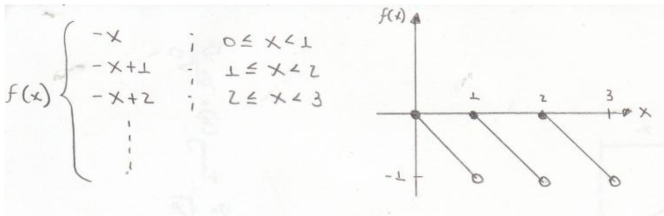
$$\mathcal{L}\left\{\int_0^t \overbrace{(1)}^{g(t-x)}\ \underbrace{\tau e^{-2\tau}Sen(\tau)}_{f(x)}\,d\tau\right\}=\frac{\frac{2(s+2)}{[(s+2)^2+1]^2}}{s}$$

$$\mathcal{L}\left\{e^{-2t}\int_0^t \tau e^{-2\tau}Sen(\tau)d\tau\right\}=\frac{\frac{2[(s+2)+2]}{\left[\left((s+2)+2\right)^2+1\right]^2}}{(s+2)}$$

$$\boxed{\boldsymbol{\mathcal{L}}\left\{\boldsymbol{e^{-2t}}\int_{\mathbf{0}}^{\boldsymbol{t}}\boldsymbol{\tau e^{-2\tau}Sen(\tau)d\tau}\right\}=\frac{\mathbf{2}(\boldsymbol{s}+\mathbf{4})(\boldsymbol{s}+\mathbf{2})}{\left[\left((\boldsymbol{s}+\mathbf{2})+\mathbf{2}\right)^{\mathbf{2}}+\mathbf{1}\right]^{\mathbf{2}}}}$$

$10)\mathcal{L}\{\|x\|-x\}$

El gráfico correspondiente a esta función es:



$T = 1$

$$\mathcal{L}\{\|x\|-x\} = \frac{1}{1-e^{-s}}\int_0^1 e^{-st}(-t)dt$$

$u = -t \quad \Rightarrow \quad du = -dt$

$dv = e^{-st}dt \quad \Rightarrow \quad v = -\frac{1}{s}e^{-st}$

$$\mathcal{L}\{\|x\|-x\} = \frac{1}{1-e^{-s}}\left[\frac{t}{s}e^{-st} + \frac{1}{s}\int e^{-st}dt\right]$$

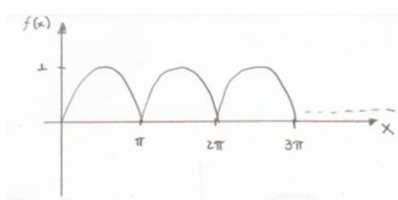
$$\mathcal{L}\{\|x\|-x\} = \frac{1}{1-e^{-s}}\left[\frac{t}{s}e^{-st} - \frac{e^{-st}}{s^2}\right]_0^1$$

$$\mathcal{L}\{\|x\|-x\} = \frac{1}{1-e^{-s}}\left[\left(\frac{1}{s}e^{-s} - \frac{e^{-s}}{s^2}\right) - \left(-\frac{1}{s^2}\right)\right]$$

$$\boxed{\boldsymbol{\mathcal{L}\{\|x\|-x\} = \frac{1}{1-e^{-s}}\left(\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}\right)}}$$

$11)\mathcal{L}\{|Sen\ x|\}$

El gráfico correspondiente a esta función es:



$T = \pi$

$$\mathcal{L}\{|Sen\ x|\} = \frac{1}{1-e^{-\pi s}}\int_0^{\pi} e^{-st}Sen(t)dt$$

$u = Sen(t) \quad \Rightarrow \quad du = -Cos(t)dt$

$dv = e^{-st}dt \quad \Rightarrow \quad v = -\frac{1}{s}e^{-st}$

$$\int e^{-st}Sen(t)dt = -\frac{Sen(t)}{s}e^{-st} - \frac{1}{s}\int e^{-st}Cos(t)dt$$

$$u = Cos(t) \quad \Rightarrow \quad du = Sen(t)dt$$

$$dv = e^{-st}dt \quad \Rightarrow \quad v = -\frac{1}{s}e^{-st}$$

$$\int e^{-st}Sen(t)dt = -\frac{Sen(t)}{s}e^{-st} - \frac{1}{s}\left[-\frac{Cos(t)}{s}e^{-st} + \frac{1}{s}\int e^{-st}Sen(t)dt\right]$$

$$\int e^{-st}Sen(t)dt = -\frac{Sen(t)}{s}e^{-st} + \frac{Cos(t)}{s^2}e^{-st} - \frac{1}{s^2}\int e^{-st}Sen(t)dt$$

$$\int e^{-st}Sen(t)dt = \frac{\left[\frac{Cos(t)}{s^2}e^{-st} - \frac{Sen(t)}{s}e^{-st}\right]}{\left(1 + \frac{1}{s^2}\right)}$$

$$\int e^{-st}Sen(t)dt = \left(\frac{s^2}{s^2+1}\right)\left[\frac{Cos(t)}{s^2}e^{-st} - \frac{Sen(t)}{s}e^{-st}\right]$$

$$\mathcal{L}\{|Sen\ x|\} = \frac{1}{1-e^{-\pi s}}\left(\frac{s^2}{s^2+1}\right)\left[\frac{Cos(t)}{s^2}e^{-st} - \frac{Sen(t)}{s}e^{-st}\right]_0^\pi$$

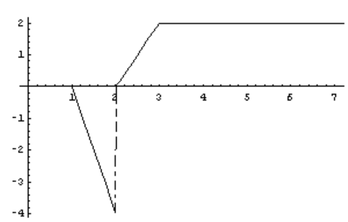
$$\mathcal{L}\{|Sen\ x|\} = \frac{1}{1-e^{-\pi s}}\left(\frac{s^2}{s^2+1}\right)\left[\left(\frac{Cos(\pi)}{s^2}e^{-s\pi} - \frac{Sen(\pi)}{s}e^{-s\pi}\right) - \left(\frac{Cos(0)}{s^2}e^{-s(0)} - \frac{Sen(0)}{s}e^{-s(0)}\right)\right]$$

$$\mathcal{L}\{|Sen\ x|\} = \frac{1}{1-e^{-\pi s}}\left(\frac{s^2}{s^2+1}\right)\left[\left(-\frac{1}{s^2}e^{-s\pi}\right) - \left(\frac{1}{s^2}\right)\right]$$

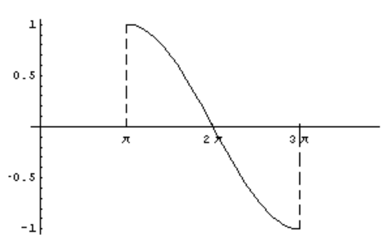
$$\boxed{\boldsymbol{\mathcal{L}\{|Sen\ x|\} = -\frac{1}{1-e^{-\pi s}}\left(\frac{s^2}{s^2+1}\right)\left(\frac{1}{s^2}e^{-s\pi} + \frac{1}{s^2}\right)}}$$

12)Encuentre la transformada de Laplace para las funciones cuyos gráficos se muestran a continuación:

a)



b)



Para a)

$P_1(1,0)\ ;\ P_2(2,-4)$

$y_1 = mx + b$

$0 = m + b\ \ ;\ -4 = 2m + b$

$y_1 = 4x + 4$

$P_1(2,0)\ ;\ P_2(3,2)$

$y_2 = mx + b$

$0 = 2m + b\ \ ;\ 2 = 3m + b$

$y_2 = 2x - 4$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{[u(t-1) - u(t-2)]y_1 + [u(t-2) - u(t-3)]y_2 + u(t-3)y_3\}$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{[u(t-1) - u(t-2)](4t+4) + [u(t-2) - u(t-3)](2t-4) + u(t-3)2\}$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{4(t+1)u(t-1) - 4u(t-2)(t+1) + 2u(t-2)(t-2) - 2u(t-3)(t-2) + 2u(t-3)\}$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{4((t+1-2)+2)u(t-1) - 4u(t-2)((t+1-3)+3) + 2u(t-2)(t-2) - 2u(t-3)((t-2-1)+1) + 2u(t-3)\}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{4(t-1)u(t-1) + 8u(t-1) - 4u(t-2)(t-2) - 12u(t-2) + 2u(t-2)(t-2) - 2u(t-3)(t-3) - 2u(t-3) + 2u(t-3)\}$$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{4(t-1)u(t-1) + 8u(t-1) - 2u(t-2)(t-2) - 12u(t-2) - 2u(t-3)(t-3)\}$

$$\boxed{\boldsymbol{\mathcal{L}\{f(t)\} = \frac{e^{-t}}{s^2} + 8e^{-t} - 2\frac{e^{-2t}}{s^2} - 12e^{-2t} - 2\frac{e^{-3t}}{s^2}}}$$

Para b)

Sabemos que el período de la función Sen(Bx) es

$T = \frac{2\pi}{B}$, *entonces* $4\pi = \frac{2\pi}{B} \quad \Rightarrow \quad B = \frac{1}{2}$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{[u(t-\pi) + u(t-3\pi)]Sen\left(\frac{x}{2}\right)\right\}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{u(t-\pi)Sen\left(\frac{x}{2}\right) + u(t-3\pi)Sen\left(\frac{x}{2}\right)\right\}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{u(t-\pi)Sen\left[\frac{1}{2}(t-\pi+\pi)\right] + u(t-3\pi)Sen\left[\frac{1}{2}(t-3\pi+3\pi)\right]\right\}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{u(t-\pi)Sen\left[\frac{1}{2}(t-\pi)+\frac{\pi}{2}\right] + u(t-3\pi)Sen\left[\frac{1}{2}(t-3\pi)+\frac{3\pi}{2}\right]\right\}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{\begin{matrix} u(t-\pi)\left[Sen\left[\frac{1}{2}(t-\pi)\right]Cos\left(\frac{\pi}{2}\right) + Cos\left[\frac{1}{2}(t-\pi)\right]Sen\left(\frac{\pi}{2}\right)\right] + \\ u(t-3\pi)\left[Sen\left[\frac{1}{2}(t-3\pi)\right]Cos\left(\frac{3\pi}{2}\right) + Cos\left[\frac{1}{2}(t-3\pi)\right]Sen\left(\frac{3\pi}{2}\right)\right] \end{matrix}\right\}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{u(t-\pi)Cos\left[\frac{1}{2}(t-\pi)\right] - u(t-3\pi)Cos\left[\frac{1}{2}(t-3\pi)\right]\right\}$$

$$\mathcal{L}\{f(t)\} = e^{-\pi s}\frac{s}{s^2+\frac{1}{4}} - e^{-3\pi s}\frac{s}{s^2+\frac{1}{4}}$$

$$\boxed{\boldsymbol{\mathcal{L}\{f(t)\} = e^{-\pi s}\frac{4s}{4s^2+1} - e^{-3\pi s}\frac{4s}{4s^2+1}}}$$

TRANSFORMADA INVERSA DE LAPLACE

1) $\boldsymbol{\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+4s+8}\right\}}$

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+4s+8}\right\}=\mathcal{L}^{-1}\left\{\frac{s+1}{(s^2+4s+4)+8-4}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+4s+8}\right\}=\mathcal{L}^{-1}\left\{\frac{s+1}{(s+2)^2+4}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+4s+8}\right\}=\mathcal{L}^{-1}\left\{\frac{(s+1+1)-1}{(s+2)^2+4}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+4s+8}\right\}=\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+4}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+4}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+4s+8}\right\}=\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+4}\right\}-\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+2)^2+4}\right\}$$

$$\boxed{\boldsymbol{\mathcal{L}^{-1}\{\frac{s+1}{s^2+4s+8}\}=e^{-2t}Cos(2t)-\frac{e^{-2t}}{2}Sen(2t)}}$$

2) $\boldsymbol{\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{(s^2+1)(s^2+4)}\right\}}$

$$\frac{1}{(s^2+1)(s^2+4)}=\frac{A(2s)+B}{s^2+1}+\frac{C(2s)+D}{s^2+4}$$

$$1=(2As+B)(s^2+4)+(2Cs+D)(s^2+1)$$

$$1=2As^3+8As+Bs^2+4B+2Cs^3+2Cs+Ds^2+D$$

$$1=(2A+2C)s^3+(B+D)s^2+(8A+2C)s+(4B+D)$$

$$0=2A+2C$$

$$0=B+D$$

$$0=8A+2C$$

$$1=4B+D$$

Resolviendo el sistema A = 0, B = 1/3, C = 0, D = -1/3

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{(s^2+1)(s^2+4)}\right\}=\mathcal{L}^{-1}\left\{e^{-2s}\left(\frac{2As+B}{s^2+1}+\frac{2Cs+D}{s^2+4}\right)\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{(s^2+1)(s^2+4)}\right\}=2A\mathcal{L}^{-1}\left\{\frac{se^{-2s}}{s^2+1}\right\}+B\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2+1}\right\}+2C\mathcal{L}^{-1}\left\{\frac{se^{-2s}}{s^2+4}\right\}+D\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2+4}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{(s^2+1)(s^2+4)}\right\}=\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2+1}\right\}-\frac{1}{6}\mathcal{L}^{-1}\left\{\frac{2e^{-2s}}{s^2+4}\right\}\quad\Rightarrow\quad\boxed{\boldsymbol{\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{(s^2+1)(s^2+4)}\right\}=u(t-2)\left[\frac{1}{3}Sen\,(t-2)-\frac{1}{6}Sen(2(t-2))\right]}}$$

$$\mathbf{3)}\boldsymbol{\mathcal{L}^{-1}\left\{ln\left(\frac{s-1}{s^2+2s+5}\right)\right\}}$$

$$\mathcal{L}\{t\,f(t)\} = -\frac{d}{ds}F(s)$$

$$t\,f(t) = \mathcal{L}^{-1}\left\{-\frac{d}{ds}F(s)\right\}$$

$$t\,f(t) = \mathcal{L}^{-1}\left\{-\frac{d}{ds}\left[ln\left(\frac{s-1}{S^2+2s+5}\right)\right]\right\}$$

$$t\,f(t) = \mathcal{L}^{-1}\left\{-\frac{d}{ds}ln(s-1)\right\} + \mathcal{L}^{-1}\left\{-\frac{d}{ds}ln[(S^2+2s+1)+5-1]\right\}$$

$$t\,f(t) = \mathcal{L}^{-1}\left\{-\frac{d}{ds}ln(s-1)\right\} + \mathcal{L}^{-1}\left\{-\frac{d}{ds}ln[(s+1)^2+4]\right\}$$

$$t\,f(t) = \mathcal{L}^{-1}\left\{-\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{-\frac{2(s+1)}{(s+1)^2+4}\right\}$$

$$t\,f(t) = -e^{t} - 2e^{-t}Cos(2t)$$

$$\boxed{\boldsymbol{f(t) = \frac{-e^{t}-2e^{-t}Cos(2t)}{t}}}$$

$$\mathbf{4)}\boldsymbol{\mathcal{L}^{-1}\left\{ln\left(\frac{s^2+9}{s^2+1}\right)\right\}}$$

$$\mathcal{L}\{t\,f(t)\} = -\frac{d}{ds}F(s)$$

$$t\,f(t) = \mathcal{L}^{-1}\left\{-\frac{d}{ds}F(s)\right\}$$

$$t\,f(t) = \mathcal{L}^{-1}\left\{-\frac{d}{ds}\left[ln\left(\frac{s^2+9}{s^2+1}\right)\right]\right\}$$

$$t\,f(t) = \mathcal{L}^{-1}\left\{-\frac{d}{ds}ln(s^2+9)\right\} + \mathcal{L}^{-1}\left\{\frac{d}{ds}ln(s^2+1)\right\}$$

$$t\,f(t) = \mathcal{L}^{-1}\left\{-\frac{2s}{s^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{2s}{s^2+1}\right\}$$

$$t\,f(t) = -2\,Cos(3t) + 2\,Cos(t)$$

$$\boxed{\boldsymbol{f(t) = \frac{2\,Cos(t)-2\,Cos(3t)}{t}}}$$

$$\mathbf{5)}\boldsymbol{\mathcal{L}^{-1}\left\{\frac{s^3+3s^2+1}{s^2(s^2+2s+2)}\right\}}$$

$$\frac{s^3+3s^2+1}{s^2(s^2+2s+2)}=\frac{A}{s}+\frac{B}{s^2}+\frac{C(2s+2)+D}{(s^2+2s+2)}$$

$$s^3+3s^2+1=As(s^2+2s+2)+B(s^2+2s+2)+2Cs(s^2)+2C(s^2)+D(s^2)$$

$$s^3+3s^2+1=As^3+2As^2+2As+Bs^2+2Bs+2B+2Cs^3+2Cs^2+Ds^2$$

$$s^3+3s^2+1=(A+2C)s^3+(2A+B+2C+D)s^2+(2A+2B)s+2B$$

$$1=A+2C$$

$$3=2A+B+2C+D$$

$$0=2A+2B$$

$$1=2B$$

Resolviendo el sistema   A = -1/2, B = ½, C = ¾, D = 2

$$\mathcal{L}^{-1}\left\{\frac{s^3+3s^2+1}{s^2(s^2+2s+2)}\right\}=A\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}+B\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}+2C\mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2+1}\right\}+D\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{s^3+3s^2+1}{s^2(s^2+2s+2)}\right\}=A+Bt+2Ce^{-t}Cos(t)+De^{-t}\,Sen(t)$$

$$\boxed{\boldsymbol{\mathcal{L}^{-1}\left\{\frac{s^3+3s^2+1}{s^2(s^2+2s+2)}\right\}=-\frac{1}{2}+\frac{t}{2}+\frac{3e^{-t}}{2}Cos(t)+2e^{-t}\,Sen(t)}}$$

$$\mathbf{6)}\boldsymbol{\mathcal{L}^{-1}\left\{\frac{2s}{(s^2+1)^3}\right\}}$$

$$\mathcal{L}^{-1}\left\{\int_s^{+\infty}F(\sigma)d\sigma\right\}=\frac{f(t)}{t}$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\}=\int_s^{+\infty}F(\sigma)d\sigma$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\}=\lim_{a\to+\infty}\int_s^{a}\frac{2s}{(s^2+1)^3}ds$$

$$u=s^2+1\quad\Rightarrow\quad du=2s$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\}=\lim_{a\to+\infty}\int_s^{a}\frac{du}{(u)^3}$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\}=-\lim_{a\to+\infty}\frac{1}{2u^2}\Big|_s^a$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = -\frac{1}{2}\lim_{a\to+\infty}\left[\frac{1}{(a^2+1)^2} - \frac{1}{(s^2+1)^2}\right]$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \frac{1}{2(s^2+1)^2}$$

$$\frac{f(t)}{t} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$$

$$\frac{f(t)}{t} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1} * \frac{1}{s^2+1}\right\}$$

$$\frac{f(t)}{t} = \frac{1}{2}\left[\int_0^t Sen(\tau)\,Sen(x-\tau)d\tau\right]$$

①$Cos(a+b) = Cos(a)Cos(b) - Sen(a)Sen(b)$

②$Cos(a-b) = Cos(a)Cos(b) + Sen(a)Sen(b)$

Multiplicando por (-1) la primera ecuación

① $-Cos(a+b) = -Cos(a)Cos(b) + Sen(a)Sen(b)$

②$Cos(a-b) = Cos(a)Cos(b) + Sen(a)Sen(b)$

Entonces ① + ②

$$Sen(a)Sen(b) = \frac{Cos(a-b) - Cos(a+b)}{2}$$

$$\frac{f(t)}{t} = \frac{1}{4}\left[\int_0^t [Cos(\tau - t + \tau) - Cos(\tau + t - \tau)]d\tau\right]$$

$$\frac{f(t)}{t} = \frac{1}{4}\left[\int_0^t [Cos(2\tau - t) - Cos(t)]d\tau\right]$$

$$\frac{f(t)}{t} = \frac{1}{4}\left[\int_0^t [Cos(2\tau - t) - Cos(t)]d\tau\right]$$

$$\frac{f(t)}{t} = \frac{1}{4}\left[\frac{1}{2}Sen(2\tau - t) - \tau\,Cos(t)\right]\begin{matrix} t \\ 0 \end{matrix}$$

$$\frac{f(t)}{t} = \frac{1}{4}\left[\frac{1}{2}Sen(2t - t) - t\,Cos(t) - \frac{1}{2}Sen(-t)\right]$$

$$f(t) = \frac{t}{4}\left[\frac{1}{2}Sen(t) - t\,Cos(t) + \frac{1}{2}Sen(t)\right]$$

$$\boxed{\boldsymbol{f(t) = \frac{t}{4}[Sen(t) - t\,Cos(t)]}}$$

$$\mathbf{7)}\boldsymbol{\mathcal{L}^{-1}\left\{\frac{\pi}{2}-Arctan\left(\frac{s}{2}\right)\right\}}$$

$$t\,f(t)=\mathcal{L}^{-1}\left\{-\frac{d}{ds}\left[\frac{\pi}{2}-Arctan\left(\frac{s}{2}\right)\right]\right\}$$

$$t\,f(t)=\mathcal{L}^{-1}\left\{-\frac{1}{1+\left(\frac{s}{2}\right)^2}\right\}$$

$$t\,f(t)=\mathcal{L}^{-1}\left\{-\frac{4}{4+s^2}\right\}$$

$$t\,f(t)=-2\mathcal{L}^{-1}\left\{\frac{2}{4+s^2}\right\}$$

$$t\,f(t)=-2\,Sen(2t)$$

$$\boxed{\boldsymbol{f(t)=\frac{-2\,Sen(2t)}{t}}}$$

$$\mathbf{8)}\boldsymbol{\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4s+5)}\right\}}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4s+5)}\right\}=\mathcal{L}^{-1}\left\{\underbrace{\frac{1}{s}}_{1}*\overbrace{\frac{1}{(s+2)^2+1}}^{e^{-2t}Sen(t)}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4s+5)}\right\}=\int_0^t e^{-2x}Sen(x)dx$$

$$\int e^{-2x}Sen(x)dx$$

$$u=Sen(x)\quad\Rightarrow\quad du=Cos(x)$$

$$dv=e^{-2x}dx\quad\Rightarrow\quad v=-\frac{1}{2}e^{-2x}$$

$$\int e^{-2x}Sen(x)dx=-\frac{Sen(x)}{2}e^{-2x}+\frac{1}{2}\int e^{-2x}Cos(x)\,dx$$

$$u=Cos(x)\quad\Rightarrow\quad du=-Sen(x)$$

$$dv=e^{-2x}dx\quad\Rightarrow\quad v=-\frac{1}{2}e^{-2x}$$

$$\int e^{-2x}Sen(x)dx=-\frac{Sen(x)}{2}e^{-2x}+\frac{1}{2}\left[-\frac{Cos(x)}{2}e^{-2x}-\frac{1}{2}\int e^{-2x}Sen(x)dx\right]$$

$$\int e^{-2x}Sen(x)dx=-\frac{Sen(x)}{2}e^{-2x}-\frac{Cos(x)}{4}e^{-2x}-\frac{1}{4}\int e^{-2x}Sen(x)dx$$

$$\int e^{-2x} Sen(x)dx = -\frac{4}{5}e^{-2x}\left[\frac{Sen(x)}{2}+\frac{Cos(x)}{4}\right]$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4s+5)}\right\} = -\frac{4}{5}e^{-2x}\left[\frac{Sen(x)}{2}+\frac{Cos(x)}{4}\right]\Bigg|_{0}^{t}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4s+5)}\right\} = -\frac{4}{5}e^{-2t}\left[\frac{Sen(t)}{2}+\frac{Cos(t)}{4}\right]+\frac{4}{5}\left[\frac{1}{4}\right]$$

$$\boxed{\boldsymbol{\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4s+5)}\right\} = -\frac{1}{5}\left[2e^{-2t}Sen(t)+e^{-2t}Cos(t)-1\right]}}$$

RESOLUCIÓN DE ECUACIONES DIFERENCIALES MEDIANTE LA TRANSFORMADA DE LAPLACE

1) $\boldsymbol{y'' - 6y' + 9y = t^2 e^{3t}}\,, \qquad \boldsymbol{y(0) = 2\,;\; y'(0) = 6}$

$\mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = \mathcal{L}\{t^2 e^{3t}\}$

$\left[s^2 Y - s\,y(0) - y'(0)\right] - 6[sY - y(0)] + 9Y = \dfrac{2!}{(s-3)^3}$

$Ys^2 - 2s - 6 - 6Ys + 12 + 9Y = \dfrac{2}{(s-3)^3}$

$Y[s^2 - 6s + 9] = \dfrac{2}{(s-3)^3} + 2s - 6$

$\mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\dfrac{2}{(s-3)^5} + \dfrac{2s}{(s-3)^2} - \dfrac{6}{(s-3)^2}\right\}$

$\mathcal{L}^{-1}\{Y\} = \dfrac{2}{4!}\mathcal{L}^{-1}\left\{\dfrac{4!}{(s-3)^5}\right\} + 2\mathcal{L}^{-1}\left\{\dfrac{s}{(s-3)^2}\right\} - 6\mathcal{L}^{-1}\left\{\dfrac{1}{(s-3)^2}\right\}$

$\dfrac{s}{(s-3)^2} = \dfrac{A}{s-3} + \dfrac{B}{(s-3)^2}$

$s = A(s-3)^2 + B(s-3) = A(s^2 - 6s + 9) + Bs - 3B = As^2 - 6As + 9A + Bs - 3B$

$s = As^2 + (B - 6A)s + (9A - 3B)$

$0 = A$

$1 = B - 6A$

$0 = 9A - 3B$

Podemos notar que el sistema no tiene solución, entonces este método no funciona, pero sabemos que

$t\,f(t) = \mathcal{L}^{-1}\left\{-\dfrac{d}{ds}F(s)\right\}$

$t\,f(t) = \mathcal{L}^{-1}\left\{-\dfrac{d}{ds}\left[\dfrac{s}{(s-3)^2}\right]\right\}$

$t\,f(t) = \mathcal{L}^{-1}\left\{-\dfrac{1}{(s-3)^2} + \dfrac{2s}{(s-3)^3}\right\}$

$t\,f(t) = -\mathcal{L}^{-1}\left\{\dfrac{1}{(s-3)^2}\right\} + 2\mathcal{L}^{-1}\left\{\dfrac{s}{(s-3)^3}\right\}$

$\dfrac{s}{(s-3)^3} = \dfrac{A}{s-3} + \dfrac{B}{(s-3)^2} + \dfrac{C}{(s-3)^3}$

$s = A(s-3)^2 + B(s-3) + C = A(s^2 - 6s + 9) + Bs - 3B + C = As^2 - 6As + 9A + Bs - 3B + C$

$s = As^2 + (B - 6A)s + (9A - 3B + C)$

$0 = A$

$1 = B - 6A$

$0 = 9A - 3B + C$

Resolviendo el sistema A = 0, B = 1, C = 3

$$t\,f(t) = -\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{A}{s-3} + \frac{B}{(s-3)^2} + \frac{C}{(s-3)^3}\right\}$$

$$t\,f(t) = -\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{B}{(s-3)^2} + \frac{C}{(s-3)^3}\right\}$$

$$t\,f(t) = -\frac{\mathcal{L}^{-1}}{1!}\left\{\frac{1!}{(s-3)^2}\right\} + \frac{2B\mathcal{L}^{-1}}{1!}\left\{\frac{1!}{(s-3)^2}\right\} + \frac{2C\mathcal{L}^{-1}}{2!}\left\{\frac{2!}{(s-3)^3}\right\}$$

$$t\,f(t) = -te^{3t} + 2te^{3t} + 3t^2e^{3t}$$

$$f(t) = -e^{3t} + 2e^{3t} + 36te^{3t}$$

$$y(t) = \frac{t^4e^{3t}}{12} + 2e^{3t} + 72te^{3t} - 6te^{3t}$$

$$\boxed{\boldsymbol{y(t) = e^{3t}\left[\frac{t^4}{12} + 2 + 66t\right]}}$$

2) $\boldsymbol{y'' + 4y = u\left(t - \frac{\pi}{4}\right) Sen\,(t)\,, \qquad y(0) = 1\,;\; y'(0) = 0}$

$$y'' + 4y = u\left(t - \frac{\pi}{4}\right) Sen\,\left[\left(t - \frac{\pi}{4}\right) + \frac{\pi}{4}\right]$$

$$y'' + 4y = u\left(t - \frac{\pi}{4}\right)\left[Sen\,\left(t - \frac{\pi}{4}\right) Cos\left(\frac{\pi}{4}\right) + Sen\left(\frac{\pi}{4}\right) Cos\left(t - \frac{\pi}{4}\right)\right]$$

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\left\{u\left(t - \frac{\pi}{4}\right)\left[Sen\,\left(t - \frac{\pi}{4}\right) Cos\left(\frac{\pi}{4}\right) + Sen\left(\frac{\pi}{4}\right) Cos\left(t - \frac{\pi}{4}\right)\right]\right\}$$

$$\left[s^2Y - s\,y(0) - y'(0)\right] + 4Y = e^{-\frac{\pi}{4}s}\left[\frac{\sqrt{2}}{2}\left(\frac{1}{s^2+1}\right) + \frac{\sqrt{2}}{2}\left(\frac{s}{s^2+1}\right)\right]$$

$$s^2Y - s + 4Y = e^{-\frac{\pi}{4}s}\left[\frac{\sqrt{2}}{2}\left(\frac{1}{s^2+1}\right) + \frac{\sqrt{2}}{2}\left(\frac{s}{s^2+1}\right)\right]$$

$$Y = \frac{e^{-\frac{\pi}{4}s}}{s^2+4}\left[\frac{\sqrt{2}}{2}\left(\frac{1}{s^2+1}\right) + \frac{\sqrt{2}}{2}\left(\frac{s}{s^2+1}\right)\right] + \frac{s}{s^2+4}$$

$$\mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{e^{-\frac{\pi}{4}s}}{s^2+4}\left[\frac{\sqrt{2}}{2}\left(\frac{1}{s^2+1}\right) + \frac{\sqrt{2}}{2}\left(\frac{s}{s^2+1}\right)\right] + \frac{s}{s^2+4}\right\}$$

$$y(t) = \frac{\sqrt{2}}{2}\mathcal{L}^{-1}\left\{\frac{e^{-\frac{\pi}{4}s}}{(s^2+4)(s^2+1)}\right\} + \frac{\sqrt{2}}{2}\mathcal{L}^{-1}\left\{\frac{s\,e^{-\frac{\pi}{4}s}}{(s^2+4)(s^2+1)}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)}\right\}$$

$$\frac{1}{(s^2+4)(s^2+1)} = \frac{A(2s)+B}{s^2+4} + \frac{C(2s)+D}{s^2+1}$$

$1 = 2As(s^2+1) + B(s^2+1) + 2Cs(s^2+4) + D(s^2+4)$

$1 = 2As^3 + 2As + Bs^2 + B + 2Cs^3 + 8Cs + Ds^2 + 4D$

$1 = (2A+2C)s^3 + (B+D)s^2 + (2A+8C)s + (B+4D)$

$0 = 2A + 2C$

$0 = B + D$

$0 = 2A + 8C$

$1 = B + 4D$

Resolviendo el sistema A = 0, B = -1/3, C = 0, D = 1/3

$$\frac{s}{(s^2+4)(s^2+1)} = \frac{A'(2s)+B'}{s^2+4} + \frac{C'(2s)+D'}{s^2+1}$$

$s = (2A'+2C')s^3 + (B'+D')s^2 + (2A'+8C')s + (B'+4D')$

$0 = 2A' + 2C'$

$0 = B' + D'$

$1 = 2A' + 8C'$

$0 = B' + 4D'$

Resolviendo el sistema A' = -1/6, B' = 0, C' = 1/6, D' = 0

$$y(t) = \frac{\sqrt{2}}{2}\mathcal{L}^{-1}\left\{e^{-\frac{\pi}{4}s}\left[\frac{2As+B}{s^2+4} + \frac{2Cs+D}{s^2+1}\right]\right\} + \frac{\sqrt{2}}{2}\mathcal{L}^{-1}\left\{e^{-\frac{\pi}{4}s}\left[\frac{2A's+B'}{s^2+4} + \frac{2C's+D'}{s^2+1}\right]\right\} + \mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)}\right\}$$

$$\begin{aligned} y(t) = &\frac{\sqrt{2}}{2}\mathcal{L}^{-1}\left\{e^{-\frac{\pi}{4}s}\left[2A\frac{s}{s^2+4} + \left(\frac{B}{2}\right)\frac{1*2}{s^2+4} + 2C\frac{s}{s^2+1} + D\frac{1}{s^2+1}\right]\right\} \\ &+ \frac{\sqrt{2}}{2}\mathcal{L}^{-1}\left\{e^{-\frac{\pi}{4}s}\left[2A'\frac{s}{s^2+4} + B'\frac{1}{s^2+4} + 2C'\frac{s}{s^2+1} + D'\frac{1}{s^2+1}\right]\right\} + \mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)}\right\} \end{aligned}$$

$$\boxed{\boldsymbol{y(t) = \frac{\sqrt{2}}{2}u\left(t-\frac{\pi}{4}\right)\left[-\frac{1}{6}Sen\left(2\left(t-\frac{\pi}{4}\right)+\frac{1}{3}Sen\left(t-\frac{\pi}{4}\right)\right)\right] + \frac{\sqrt{2}}{2}u\left(t-\frac{\pi}{4}\right)\left[-\frac{1}{3}Cos\left(2\left(t-\frac{\pi}{4}\right)+\frac{1}{3}Cos\left(t-\frac{\pi}{4}\right)\right)\right] + Cos(2t)}}$$

3) $\boldsymbol{f(t) + 4\int_0^t Sen(\tau)f(t-\tau)d\tau = 2t}$

$$\mathcal{L}\{f(t)\} + 4\mathcal{L}\left\{\int_0^t Sen(\tau)f(t-\tau)d\tau\right\} = 2\mathcal{L}\{t\}$$

$$Y + 4Y\left(\frac{1}{s^2+1}\right) = 2\left(\frac{1}{s^2}\right)$$

$$Y = \frac{2\left(\frac{1}{s^2}\right)}{\left(1 + \frac{4}{s^2+1}\right)}$$

$$Y = \frac{2(s^2+5)}{s^2(s^2+1)}$$

$$\mathcal{L}^{-1}\{Y\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + 10\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\}$$

$$y(t) = 2\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + 10\int_0^t Sen(\tau)(t-\tau)d\tau$$

$$\int Sen(\tau)(t-\tau)d\tau$$

$$u = t - \tau \quad \Rightarrow \quad du = -d\tau$$

$$dv = Sen(\tau)d\tau \quad \Rightarrow \quad v = -Cos(\tau)$$

$$= -Cos(\tau)(t-\tau) - \int Cos(\tau)d\tau$$

$$= -Cos(\tau)(t-\tau) - Sen(\tau)$$

$$y(t) = 2\,Sen(t) - 10[Cos(\tau)(t-\tau) + Sen(\tau)]\,\begin{matrix} t \\ 0 \end{matrix}$$

$$y(t) = 2\,Sen(t) - 10[Sen(t) - t]$$

$$\boxed{\boldsymbol{y(t) = 10t - 8\,Sen(t)}}$$

4) $\boldsymbol{y'' + 2y' + 2y = \delta(t-\pi), \qquad y(0) = y'(0) = 0}$

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{\delta(t-\pi)\}$$

$$\left[s^2Y - s\,y(0) - y'(0)\right] + 2[sY - y(0)] + 2Y = e^{-\pi s}$$

$$Ys^2 + 2Ys + 2Y = e^{-\pi s}$$

$$Y = \frac{e^{-\pi s}}{s^2 + s + 2}$$

$$\mathcal{L}^{-1}\{Y\} = 2\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{(s+1)^2+1}\right\}$$

$$\boxed{\boldsymbol{y(t) = 2\,u(t-\pi)e^{(t-\pi)}Sen(t-\pi)}}$$

5) $\boldsymbol{y'' + 2y' + 2y = Cos(t)\delta(t - 3\pi), \qquad y(0) = 1, \qquad y'(0) = -1}$

$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{Cos(t)\delta(t - 3\pi)\}$

$[s^2Y - s\,y(0) - y'(0)] + 2[sY - y(0)] + 2Y = Cos(3\pi)e^{-3\pi s}$

$Ys^2 - s + 1 + 2Ys - 2 + 2Y = -e^{-3\pi s}$

$Y(s^2 + 2s + 2) = -e^{-3\pi s} + (s + 1)$

$$\mathcal{L}^{-1}\{Y\} = -\mathcal{L}^{-1}\left\{\frac{e^{-3\pi s}}{s^2 + 2s + 2}\right\} + \mathcal{L}^{-1}\left\{\frac{(s + 1)}{s^2 + 2s + 2}\right\}$$

$$y(t) = -\mathcal{L}^{-1}\left\{\frac{e^{-3\pi s}}{(s + 1)^2 + 1}\right\} + \mathcal{L}^{-1}\left\{\frac{(s + 1)}{(s + 1)^2 + 1}\right\}$$

$\boxed{\boldsymbol{y(t) = -u(t - 3\pi)e^{(t-3\pi)}Sen(t - 3\pi) + e^{-t}Cos(t)}}$

6) $\boldsymbol{ty'' - ty' - y = 0 \quad , \qquad y(0) = 0 \quad , \qquad y'(0) = 3}$

$\mathcal{L}\{ty''\} - \mathcal{L}\{ty'\} - \mathcal{L}\{y\} = 0$

$$-\frac{d}{ds}[s^2Y - s\,y(0) - y'(0)] + \frac{d}{ds}[sY - y(0)] - Y = 0$$

$-2sY - s^2Y' + Y + sY' - Y = 0$

$Y'(1 - s^2) = 2sY$

$$\frac{dY}{ds}(1 - s^2) = 2sY$$

$$\frac{dY}{Y} = \frac{2s}{1 - s^2}ds \quad \Rightarrow \quad \int\frac{dY}{Y} = 2\int\frac{s}{1 - s^2}ds$$

$u = s^2 \quad \Rightarrow \quad u = 2s\,ds$

$$\int\frac{dY}{Y} = -\int\frac{1}{1 - u}ds$$

$ln|Y| = -ln|1 - u|$

$$e^{ln|Y|} = e^{-ln|1-u|} \quad \Rightarrow \quad Y = \frac{1}{1 - s^2}$$

$$\mathcal{L}^{-1}\{Y\} = -\mathcal{L}^{-1}\left\{\frac{1}{s^2 - 1}\right\}$$

$\boxed{\boldsymbol{y(t) = -Senh(t)}}$

7) $\boldsymbol{y'' - 2y' + y = e^{t} \quad , \qquad y(0) = 0 \quad , \qquad y'(1) = \frac{11}{2}e}$

No conocemos el valor de y'(0), entonces vamos a realizar un artificio, multiplicaremos por "t"

$$ty'' - 2ty' + ty = te^{t}$$

$$\mathcal{L}\{ty''\} - 2\mathcal{L}\{ty'\} + \mathcal{L}\{ty\} = \mathcal{L}\{te^{t}\}$$

$$-\frac{d}{ds}\left[s^2Y - s\,y(0) - y'(0)\right] + \frac{d}{ds}[sY - y(0)] - \frac{d}{ds}[Y] = \frac{1}{(s-1)^2}$$

$$-2sY - s^2Y' + Y + sY' - Y' = \frac{1}{(s-1)^2}$$

$$Y'(-s^2 + s - 1) + (1 - 2s)Y = \frac{1}{(s-1)^2}$$

$$Y'(s^2 - s + 1) + (2s - 1)Y = -\frac{1}{(s-1)^2}$$

$$Y' + \frac{(2s-1)}{(s^2 - s + 1)}Y = -\frac{1}{(s-1)^2(s^2 - s + 1)}$$

$$u(s) = e^{\int p(s)ds} \quad \Rightarrow \quad u(s) = e^{\int \frac{(2s-1)}{(s^2-s+1)}ds}$$

Resolviendo la integral:

$$\int \frac{(2s-1)}{(s^2 - s + 1)}ds$$

$$u = s^2 - s \quad \Rightarrow \quad du = (2s - 1)ds$$

$$\int \frac{du}{u+1} \quad \Rightarrow \quad ln|u + 1| \quad \Rightarrow \quad ln|s^2 - s + 1|$$

Entonces:

$$u(s) = e^{ln|s^2-s+1|} \quad \Rightarrow \quad u(s) = s^2 - s + 1$$

$$\frac{d}{ds}[(s^2 - s + 1)Y] = -\frac{1}{(s-1)^2}$$

$$\int d[(s^2 - s + 1)Y] = -\int \frac{1}{(s-1)^2}ds$$

$$(s^2 - s + 1)Y = \frac{1}{s-1}$$

$$\mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s^2 - s + 1)}\right\} \quad \Rightarrow \quad \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)\left[\left(s^2 - s + \frac{1}{4}\right) + 1 - \frac{1}{4}\right]}\right\}$$

$$\mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)\left[\left(s - \frac{1}{2}\right)^2 + \frac{3}{4}\right]}\right\}$$

Aplicando convolución:

$$\mathcal{L}^{-1}\left\{\underbrace{\frac{1}{(s-1)}}_{e^{t}} * \overbrace{\frac{1}{\left[\left(s-\frac{1}{2}\right)^{2}+\frac{3}{4}\right]}}^{\frac{2}{\sqrt{3}}e^{\frac{1}{2}t}\,Sen\left(\frac{\sqrt{3}}{2}t\right)}\right\}$$

$$\frac{2}{\sqrt{3}}\int_{0}^{t} e^{(t-x)}\, e^{\frac{1}{2}x}\, Sen\left(\frac{\sqrt{3}}{2}x\right)dx$$

$$\frac{2}{\sqrt{3}}e^{t}\int_{0}^{t} e^{-\frac{1}{2}x}\, Sen\left(\frac{\sqrt{3}}{2}x\right)dx$$

Integrando por partes:

$$u = e^{-\frac{1}{2}x} \quad \Rightarrow \quad du = -\frac{1}{2}e^{-\frac{1}{2}x}dx$$

$$dv = Sen\left(\frac{\sqrt{3}}{2}x\right)dx \quad \Rightarrow \quad v = \frac{2}{\sqrt{3}}Cos\left(\frac{\sqrt{3}}{2}x\right)$$

Entonces:

$$\int e^{-\frac{1}{2}x}\, Sen\left(\frac{\sqrt{3}}{2}x\right)dx = \frac{2}{\sqrt{3}}e^{-\frac{1}{2}x}Cos\left(\frac{\sqrt{3}}{2}x\right) + \frac{1}{\sqrt{3}}\int e^{-\frac{1}{2}x}Cos\left(\frac{\sqrt{3}}{2}x\right)dx$$

Integrando nuevamente por partes:

$$u = e^{-\frac{1}{2}x} \quad \Rightarrow \quad du = -\frac{1}{2}e^{-\frac{1}{2}x}dx$$

$$dv = Cos\left(\frac{\sqrt{3}}{2}x\right)dx \quad \Rightarrow \quad v = -\frac{2}{\sqrt{3}}Sen\left(\frac{\sqrt{3}}{2}x\right)$$

Entonces:

$$\int e^{-\frac{1}{2}x}\, Sen\left(\frac{\sqrt{3}}{2}x\right)dx = \frac{2}{\sqrt{3}}e^{-\frac{1}{2}x}Cos\left(\frac{\sqrt{3}}{2}x\right) + \frac{1}{\sqrt{3}}\left[-\frac{2}{\sqrt{3}}e^{-\frac{1}{2}x}Sen\left(\frac{\sqrt{3}}{2}x\right) - \frac{1}{\sqrt{3}}\int e^{-\frac{1}{2}x}Sen\left(\frac{\sqrt{3}}{2}x\right)dx\right]$$

$$\int e^{-\frac{1}{2}x}\, Sen\left(\frac{\sqrt{3}}{2}x\right)dx = \frac{2}{\sqrt{3}}e^{-\frac{1}{2}x}Cos\left(\frac{\sqrt{3}}{2}x\right) - \frac{2}{3}e^{-\frac{1}{2}x}Sen\left(\frac{\sqrt{3}}{2}x\right) - \frac{1}{3}\int e^{-\frac{1}{2}x}Sen\left(\frac{\sqrt{3}}{2}x\right)dx$$

$$\int e^{-\frac{1}{2}x}\, Sen\left(\frac{\sqrt{3}}{2}x\right)dx = \frac{3}{4}\left[\frac{2}{\sqrt{3}}e^{-\frac{1}{2}x}Cos\left(\frac{\sqrt{3}}{2}x\right) - \frac{2}{3}e^{-\frac{1}{2}x}Sen\left(\frac{\sqrt{3}}{2}x\right)\right]$$

Luego tenemos que:

$$\int_0^t e^{-\frac{1}{2}x}\, Sen\left(\frac{\sqrt{3}}{2}x\right)dx$$

$$\frac{3}{4}\left[\frac{2}{\sqrt{3}}e^{-\frac{1}{2}x}Cos\left(\frac{\sqrt{3}}{2}x\right)-\frac{2}{3}e^{-\frac{1}{2}x}Sen\left(\frac{\sqrt{3}}{2}x\right)\right]\Bigg|_0^t$$

$$\frac{3}{4}\left[\frac{2}{\sqrt{3}}e^{-\frac{1}{2}t}Cos\left(\frac{\sqrt{3}}{2}t\right)-\frac{2}{3}e^{-\frac{1}{2}t}Sen\left(\frac{\sqrt{3}}{2}t\right)-\frac{2}{\sqrt{3}}e^{-\frac{1}{2}(0)}Cos\left(\frac{\sqrt{3}}{2}(0)\right)+\frac{2}{3}e^{-\frac{1}{2}(0)}Sen\left(\frac{\sqrt{3}}{2}(0)\right)\right]$$

$$\frac{3}{4}\left[\frac{2}{\sqrt{3}}e^{-\frac{1}{2}t}Cos\left(\frac{\sqrt{3}}{2}t\right)-\frac{2}{3}e^{-\frac{1}{2}t}Sen\left(\frac{\sqrt{3}}{2}t\right)-\frac{2}{\sqrt{3}}\right]$$

Entonces:

$$y(t)=\left(\frac{2}{\sqrt{3}}\right)\frac{3}{4}e^{t}\left[\frac{2}{\sqrt{3}}e^{-\frac{1}{2}t}Cos\left(\frac{\sqrt{3}}{2}t\right)-\frac{2}{3}e^{-\frac{1}{2}t}Sen\left(\frac{\sqrt{3}}{2}t\right)-\frac{2}{\sqrt{3}}\right]$$

$$\boxed{\boldsymbol{y(t)=\frac{3}{2\sqrt{3}}e^{t}\left[\frac{2}{\sqrt{3}}e^{-\frac{1}{2}t}Cos\left(\frac{\sqrt{3}}{2}t\right)-\frac{2}{3}e^{-\frac{1}{2}t}Sen\left(\frac{\sqrt{3}}{2}t\right)-\frac{2}{\sqrt{3}}\right]}}$$

SISTEMAS DE ECUACIONES DIFERENCIALES

$$\begin{cases} \boldsymbol{x_1' = 3x_1 - x_2} \\ \boldsymbol{x_2' = 4x_1 + 3x_2} \end{cases}$$

Derivando la primera ecuación:

$x_1'' = 3x'_1 - x'_2 \quad (3)$

(2) en (3)

$x_1'' = 3x'_1 - (4x_1 + 3x_2)$

$x_1'' = 3x'_1 - 4x_1 - 3x_2 \quad (4)$

(1) en (4)

$x_1'' = 3x'_1 - 4x_1 - 3(3x_1 - x_1')$

$x_1'' = 3x'_1 - 4x_1 - 9x_1 + 3x_1'$

$x_1'' - 6x_1' + 13x_1 = 0$

Entonces:

$x_1 = e^{rt}$

$x'_1 = re^{rt}$

$x''_1 = r^2e^{rt}$

Reemplazando:

$r^2e^{rt} - 6re^{rt} + 13e^{rt} = 0$

$e^{rt}(r^2 - 6r + 13) = 0 \quad \Rightarrow \quad r^2 - 6r + 13 = 0$

$$r_{1,2} = \frac{6 \pm \sqrt{36 - 4(1)(13)}}{2} = 3 \pm 2i$$

Entonces:

$$\boxed{\boldsymbol{x_1 = e^{3t}[C_1\, Cos(2t) + C_2\, Sen(2t)]}}$$

Pero:

$x_2 = 3x_1 - x_1'$

$$\boxed{\boldsymbol{x_2 = 3e^{3t}[C_1\, Cos(2t) + C_2\, Sen(2t)] - 3e^{3t}[C_1\, Cos(2t) + C_2\, Sen(2t)] - e^{3t}[2C_2\, Cos(2t) - 2C_1\, Sen(2t)]}}$$

OPERADORES DIFERENCIALES

$$\mathbf{1)}\begin{cases} \boldsymbol{x_1^{'} = x_1 + x_2} \\ \boldsymbol{x_2^{'} = 4x_1 - 2x_2} \end{cases}$$

$x_1^{'} = Dx_1$; $x_1^{''} = D^2x_1$

Entonces:

$Dx_1 = x_1 + x_2$; $Dx_2 = 4x_1 - 2x_2$

Luego:

$(D-1)x_1 - x_2 = 0 \ \ (1)$

$-4x_1 + (D+2)x_2 = 0 \ \ (2)$

Multiplicando por 4 a (1) y por (D+2) a (2), y luego sumamos (1)+(2):

$-4x_2 + (D-1)(D+2)x_2 = 0$

$-4x_2 + (D^2 - 3D + 2)x_2 = 0$

$-4x_2 + x''_2 - 3x'_2 + 2x_2 = 0$

$x''_2 - 3x'_2 - 2x_2 = 0$

Entonces:

$x_2 = e^{rt}$

$x'_2 = re^{rt}$

$x''_2 = r^2e^{rt}$

Reemplazando:

$r^2e^{rt} - 3re^{rt} - 2e^{rt} = 0$

$e^{rt}(r^2 - 3r - 2) = 0 \quad \Rightarrow \quad r^2 - 3r - 2 = 0$

$$r_{1,2} = \frac{3 \pm \sqrt{9 - 4(1)(-2)}}{2} = \frac{3 \pm \sqrt{17}}{2}$$

Entonces:

$$\boxed{\boldsymbol{x_2 = C_1 e^{\frac{3+\sqrt{17}}{2}x} + C_2 e^{\frac{3-\sqrt{17}}{2}x}}}$$

Pero:

$$x_1 = \frac{1}{4}(x_2^{'} + 2x_2)$$

$$\boxed{\boldsymbol{x_1 = \frac{1}{4}\left[\left(\frac{3+\sqrt{17}}{2}\right)C_1 e^{\frac{3+\sqrt{17}}{2}x} + \left(\frac{3-\sqrt{17}}{2}\right)C_2 e^{\frac{3-\sqrt{17}}{2}x}\right] + 2\left[C_1 e^{\frac{3+\sqrt{17}}{2}x} + C_2 e^{\frac{3-\sqrt{17}}{2}x}\right]}}$$

$$\mathbf{2)}\begin{cases} \boldsymbol{x^{'} = 2x - 3y + 2\,Sen(2t)} \\ \boldsymbol{y^{'} = x - 2y - Cos(2t)} \end{cases}$$

$Dx = 2x - 3y + 2\,Sen(2t)$

$Dy = x - 2y - Cos(2t)$

Luego:

$(D-2)x + 3y = 2\,Sen(2t)\ \ (1)$

$x - (D-2)y + 3y = Cos(2t)\ \ \ (2)$

Multiplicando por –(D+2) a (2), y luego sumamos (1)+(2):

$3y + (D-2)(D+2)y = 2\,Sen(2t) + Cos(2t)$

$3y + (D^2-4)y = 2\,Sen(2t) + 2\,Sen(2t) + 2\,Cos(2t)$

Entonces:

$y^{''} - y = 4\,Sen(2t) + 2\,Cos(2t)$

<u>Encontrando la solución complementaria:</u>

$y^{''} - y = 0$

Luego:

$y = e^{rt}$

$y' = re^{rt}$

$y'' = r^2e^{rt}$

Reemplazando:

$r^2e^{rt} - e^{rt} = 0 \quad \Rightarrow \quad e^{rt}(r^2-1) = 0$

$r_{1,2} = \pm 1$

Entonces:

$y_c = C_1e^t + C_2e^{-t}$

$\therefore C.F.S = \{e^t\ , e^{-t}\}$

<u>Encontrando la solución particular:</u>

$y_p = A\,Cos(2t) + B\,Sen(2t)$

$y^{'}{}_p = -2A\,Sen(2t) + 2B\,Cos(2t)$

$y''{}_p = -4A\,Cos(2t) - 4B\,Sen(2t)$

Reemplazando:

$-4A\ Cos(2t) - 4B\ Sen(2t) - A\ Cos(2t) - B\ Sen(2t) = 4\ Sen(2t) + 2\ Cos(2t)$

$-5A\ Cos(2t) - 5B\ Sen(2t) = 4\ Sen(2t) + 2\ Cos(2t)$

$-5A = 2 \quad \Rightarrow \quad A = -2/5$

$-5B = 4 \quad \Rightarrow \quad B = -4/5$

$y_p = -\frac{2}{5}Cos(2t) - \frac{4}{5}\ Sen(2t)$

Entonces:

$$\boxed{\boldsymbol{y(t) = C_1 e^t + C_2 e^{-t} - \frac{2}{5}Cos(2t) - \frac{4}{5}\ Sen(2t)}}$$

Pero:

$x(t) = y' + 2y + Cos(2t)$

$x(t) = C_1 e^t - C_2 e^{-t} + \frac{4}{5}Sen(2t) - \frac{8}{5}\ Cos(2t) + 2\left[C_1 e^t + C_2 e^{-t} - \frac{2}{5}Cos(2t) - \frac{4}{5}\ Sen(2t)\right] + Cos(2t)$

$x(t) = C_1 e^t - C_2 e^{-t} + \frac{4}{5}Sen(2t) - \frac{8}{5}\ Cos(2t) + 2C_1 e^t + 2C_2 e^{-t} - \frac{4}{5}Cos(2t) - \frac{8}{5}\ Sen(2t) + Cos(2t)$

$$\boxed{\boldsymbol{x(t) = 3C_1 e^t + C_2 e^{-t} - \frac{4}{5}Sen(2t) - \frac{11}{5}\ Cos(2t)}}$$

VALORES Y VECTORES PROPIOS

$$\mathbf{1)}\ \boldsymbol{X}' = \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} \end{pmatrix} \boldsymbol{X}$$

$det(A - \lambda I) = 0$

$$\begin{vmatrix} 0-\lambda & 1 & 1 \\ 1 & 0-\lambda & 1 \\ 1 & 1 & 0-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$$

$$-\lambda \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 1 & -\lambda \end{vmatrix} + 1 \begin{vmatrix} 1 & -\lambda \\ 1 & 1 \end{vmatrix} = 0$$

$-\lambda(\lambda^2 - 1) - (-\lambda - 1) + (1 + \lambda) = 0$

$-\lambda(\lambda - 1)(\lambda + 1) + (\lambda + 1) + (\lambda + 1) = 0$

$(\lambda + 1)[-\lambda(\lambda - 1) + (\lambda + 1)] = 0$

$-(\lambda + 1)(\lambda^2 - \lambda - 2) = 0$

$-(\lambda + 1)(\lambda + 1)(\lambda - 2) = 0 \quad \Rightarrow \quad \lambda_1 = -1 \ ; \ \lambda_2 = -1 \ ; \ \lambda_3 = 2$

Entonces:

Para $\lambda_1 = -1$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \Rightarrow \quad a = -b - c$$

$$\varepsilon_{\lambda=-1} = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \middle| \, a = -b - c \right\} \quad \Rightarrow \quad \beta_{\varepsilon_{\lambda=-1}} = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Para $\lambda_3 = 2$

$$\left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \Rightarrow \quad \begin{array}{c} -2a + b + c = 0 \\ b = c \end{array} \quad \rightarrow \quad a = c$$

$$\varepsilon_{\lambda=2} = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \middle| \, b = c \ ; \ a = c \ ; \ c \in \mathbb{R} \right\} \quad \Rightarrow \quad \beta_{\varepsilon_{\lambda=2}} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Finalmente:

$$\boxed{\boldsymbol{x} = \boldsymbol{C_1} \begin{pmatrix} \mathbf{-1} \\ \mathbf{1} \\ \mathbf{0} \end{pmatrix} \boldsymbol{e^{-t}} + \boldsymbol{C_2} \begin{pmatrix} \mathbf{-1} \\ \mathbf{0} \\ \mathbf{1} \end{pmatrix} \boldsymbol{e^{-t}} + \boldsymbol{C_3} \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \end{pmatrix} \boldsymbol{e^{-2t}}}$$

$$\mathbf{2)}\ \boldsymbol{X}' = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{2} & \mathbf{1} & \mathbf{-2} \\ \mathbf{3} & \mathbf{2} & \mathbf{1} \end{pmatrix} \boldsymbol{X} \quad ; \quad \boldsymbol{x}(\mathbf{0}) = \begin{pmatrix} \mathbf{1} \\ \mathbf{-1} \\ \mathbf{0} \end{pmatrix}$$

$det(A - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 3 & 2 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)[(1-\lambda)^2 + 4] = 0$$

$$(1-\lambda)(1 - 2\lambda + \lambda^2 + 4) = 0$$

$$(1-\lambda)(\lambda^2 - 2\lambda + 5) = 0$$

$$\lambda_1 = 1 \quad ; \quad \lambda_{2,3} = \frac{2 \pm \sqrt{4 - 4(2)(5)}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

Entonces:

Para $\lambda_1 = 1$

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 \\ 3 & 2 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{c} 2a - 2c = 0 \quad \Rightarrow \quad a = c \\ 3a + 2b = 0 \quad \Rightarrow \quad b = -\frac{3}{2}a \end{array}$$

$$\varepsilon_{\lambda=-1} = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \middle| a = c \ ; \ b = -\frac{3}{2}a \ ; \ a \in \mathbb{R} \right\} \Rightarrow \beta_{\varepsilon_{\lambda=-1}} = \left\{ \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} \right\}$$

Para $\lambda_2 = 1 + 2i$

$$\left(\begin{array}{ccc|c} -2i & 0 & 0 & 0 \\ 2 & -2i & -2 & 0 \\ 3 & 2 & -2i & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} -2i & 0 & 0 & 0 \\ 2 & -2i & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{c} -2ia = 0 \quad \Rightarrow \quad a = 0 \\ 2a - 2ib - 2c = 0 \quad \Rightarrow \quad c = -ib \end{array}$$

$$\varepsilon_{\lambda=1+2i} = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \middle| a = 0 \ ; \ c = -ib \ ; \ b \in \mathbb{R} \right\} \Rightarrow \beta_{\varepsilon_{\lambda=1+2i}} = \left\{ \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix} \right\}$$

Para $\lambda_3 = 1 - 2i$

Es la conjugada de la segunda base, entonces:

$$\beta_{\varepsilon_{\lambda=1-2i}} = \left\{ \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix} \right\}$$

Entonces:

$$x = C_1\begin{pmatrix}2\\-3\\2\end{pmatrix}e^{t} + C_2\begin{pmatrix}0\\1\\-i\end{pmatrix}e^{(1+2i)t} + C_3\begin{pmatrix}0\\1\\i\end{pmatrix}e^{(1-2i)t}$$

$$x = C_1\begin{pmatrix}2\\-3\\2\end{pmatrix}e^{t} + e^{t}\left[C_2\begin{pmatrix}0\\1\\-i\end{pmatrix}e^{2it} + C_3\begin{pmatrix}0\\1\\i\end{pmatrix}e^{-2it}\right]$$

$$x = C_1\begin{pmatrix}2\\-3\\2\end{pmatrix}e^{t} + e^{t}\left[C_2\left[\begin{pmatrix}0\\1\\0\end{pmatrix} + i\begin{pmatrix}0\\0\\-1\end{pmatrix}\right](Cos\ 2t + i\ Sen\ 2t) + C_3\left[\begin{pmatrix}0\\1\\0\end{pmatrix} + i\begin{pmatrix}0\\0\\1\end{pmatrix}\right](Cos\ 2t - i\ Sen\ 2t)\right]$$

Ahora, solo desarrollemos:

$$x = C_1\begin{pmatrix}2\\-3\\2\end{pmatrix}e^{t} + e^{t}\left[C_2\left[\begin{pmatrix}0\\1\\0\end{pmatrix} + i\begin{pmatrix}0\\0\\-1\end{pmatrix}\right](Cos\ 2t + i\ Sen\ 2t)\right]$$

$$x = C_1\begin{pmatrix}2\\-3\\2\end{pmatrix}e^{t} + e^{t}\left[C_2\begin{pmatrix}0\\1\\0\end{pmatrix}Cos\ 2t + C_2 i\begin{pmatrix}0\\1\\0\end{pmatrix}Sen\ 2t + C_2 i\begin{pmatrix}0\\0\\-1\end{pmatrix}Cos\ 2t + C_2 i^2\begin{pmatrix}0\\0\\-1\end{pmatrix}Sen\ 2t\right]$$

$$x = C_1\begin{pmatrix}2\\-3\\2\end{pmatrix}e^{t} + e^{t}\left[C_2\left[\begin{pmatrix}0\\1\\0\end{pmatrix}Cos\ 2t - \begin{pmatrix}0\\0\\-1\end{pmatrix}Sen\ 2t\right] + C_2 i\left[\begin{pmatrix}0\\1\\0\end{pmatrix}Sen\ 2t + \begin{pmatrix}0\\0\\-1\end{pmatrix}Cos\ 2t\right]\right]$$

$$x = C_1\begin{pmatrix}2\\-3\\2\end{pmatrix}e^{t} + e^{t}\left[C_2\left[\begin{pmatrix}0\\1\\0\end{pmatrix}Cos\ 2t + \begin{pmatrix}0\\0\\1\end{pmatrix}Sen\ 2t\right] + C_3\left[\begin{pmatrix}0\\1\\0\end{pmatrix}Sen\ 2t - \begin{pmatrix}0\\0\\1\end{pmatrix}Cos\ 2t\right]\right]$$

$$x = C_1\begin{pmatrix}2\\-3\\2\end{pmatrix}e^{t} + e^{t}\left[C_2\left[\begin{pmatrix}0\\1\\0\end{pmatrix}Cos\ 2t + \begin{pmatrix}0\\0\\1\end{pmatrix}Sen\ 2t\right] + C_3\left[\begin{pmatrix}0\\1\\0\end{pmatrix}Sen\ 2t - \begin{pmatrix}0\\0\\1\end{pmatrix}Cos\ 2t\right]\right]$$

Sabemos que $x(0) = \begin{pmatrix}1\\-1\\0\end{pmatrix}$

$$\begin{pmatrix}1\\-1\\0\end{pmatrix} = C_1\begin{pmatrix}2\\-3\\2\end{pmatrix} + C_2\begin{pmatrix}0\\1\\0\end{pmatrix} - C_3\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}2C_1\\-3C_1 + C_2\\2C_1 - C_3\end{pmatrix}$$

Resolviendo el sistema:

$$C_1 = \frac{1}{2} \quad ; \quad C_2 = \frac{1}{2} \quad ; \quad C_3 = 1$$

Finalmente:

$$\boxed{\boldsymbol{x = \frac{1}{2}\begin{pmatrix}2\\-3\\2\end{pmatrix}e^{t} + e^{t}\left[\frac{1}{2}\left[\begin{pmatrix}0\\1\\0\end{pmatrix}Cos\ 2t + \begin{pmatrix}0\\0\\1\end{pmatrix}Sen\ 2t\right] + \left[\begin{pmatrix}0\\1\\0\end{pmatrix}Sen\ 2t - \begin{pmatrix}0\\0\\1\end{pmatrix}Cos\ 2t\right]\right]}}$$

3) $\boldsymbol{X'} = \begin{pmatrix} \boldsymbol{2} & \boldsymbol{1} & \boldsymbol{6} \\ \boldsymbol{0} & \boldsymbol{2} & \boldsymbol{5} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{2} \end{pmatrix} \boldsymbol{X}$

$det(A - \lambda I) = 0$

$$\begin{vmatrix} 2-\lambda & 1 & 6 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\begin{vmatrix} 2-\lambda & 5 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(2-\lambda)(2-\lambda) = 0$$

$$(2-\lambda)^3 = 0$$

Cuando una matriz A solo tiene un vector propio asociado con un valor λ_1 de multiplicidad m, se puede determinar las soluciones de la siguiente forma:

$$x_m = K_{m1}\frac{t^{m-1}}{(m-1)!}e^{\lambda_1 t} + K_{m2}\frac{t^{m-2}}{(m-2)!}e^{\lambda_1 t} + \ldots\ldots\ldots + K_{mm}\, e^{\lambda_1 t}$$

En que K_{ij} son vectores columnas

Para nuestro caso la tercera solución se la determina de la siguiente manera:

$$x_3 = K\frac{t^2}{2}e^{\lambda_1 t} + P\, te^{\lambda_1 t} + Q\, e^{\lambda_1 t}$$

En donde:

$$K = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} \ , \ P = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix} \ , \ Q = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}$$

Al sustituir en el sistema X' = AX , los vectores columnas K, P, Q deben cumplir con:

$$(A - \lambda_1 I)K = 0$$
$$(A - \lambda_1 I)P = K$$
$$(A - \lambda_1 I)Q = P$$

La ecuación característica $(2-\lambda)^3 = 0$ indica que $\lambda_1 = 2$ es un valor de multiplicidad tres y al resolver tenemos:

Para $\lambda_1 = 2$

$$\left(\begin{array}{ccc|c} 0 & 1 & 6 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{c} b + 6c = 0 \quad \Rightarrow \quad b = 0 \\ 5c = 0 \quad \Rightarrow \quad c = 0 \end{array}$$

$$\varepsilon_{\lambda=2} = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \middle| \, b = 0 \ ; \ c = 0 \ ; \ a \,\epsilon\, \mathbb{R} \right\} \quad \Rightarrow \quad \beta_{\varepsilon_{\lambda=2}} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

Entonces:

$$K = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Luego resolvemos los sistemas:

1er sistema

$$(A - \lambda_1 I)P = K$$

$$\left(\begin{pmatrix} 2 & 1 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} p_2 + 6p_3 \\ 5p_3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} p_2 + 6p_3 = 1 \\ 5p_3 = 0 \\ 0 = 0 \end{matrix}$$

Resolviendo tenemos que:

$$P = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \quad \Rightarrow \quad P = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

2do sistema

$$(A - \lambda_1 I)Q = P$$

$$\left(\begin{pmatrix} 2 & 1 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} q_2 + 6q_3 \\ 5q_3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} q_2 + 6q_3 = 0 \\ 5q_3 = 1 \\ 0 = 0 \end{matrix}$$

Resolviendo tenemos que:

$$Q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \quad \Rightarrow \quad Q = \begin{pmatrix} 0 \\ -6/5 \\ 1/5 \end{pmatrix}$$

Finalmente las soluciones del sistema de ecuaciones diferenciales es:

$$x = C_1 K e^{2t} + C_2 [K\ te^{2t} + P\ e^{2t}] + C_3 \left[K\ \frac{t^2}{2} e^{2t} + P\ te^{2t} + Q\ e^{2t} \right]$$

$$\boxed{\boldsymbol{x = C_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{2t} + C_2 \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{2t} \right] + C_3 \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \frac{t^2}{2} e^{2t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ -6/5 \\ 1/5 \end{pmatrix} e^{2t} \right]}}$$

RESOLUCIÓN DE SISTEMAS DE ECUACIONES DIFERENCIALES UTILIZANDO TRANSFORMADA DE LAPLACE

$$\mathbf{1)}\begin{cases} \boldsymbol{x' + 2x + 6\int_0^t y(u)du = -2} \\ \boldsymbol{x' + y' + y = 0} \\ \boldsymbol{x(0) = -5\ ;\ \ y(0) = 6} \end{cases}$$

Aplicando transformada de Laplace a cada ecuación:

$$\mathcal{L}\{x'\} + 2\mathcal{L}\{x\} + 6\mathcal{L}\left\{\int_0^t y(u)du\right\} = -2\mathcal{L}\{1\}$$

$$\mathcal{L}\{x'\} + \mathcal{L}\{y'\} + \mathcal{L}\{y\} = 0$$

$$[sX - x(0)] + 2X + 6\frac{Y}{s} = -\frac{2}{s} \quad \Rightarrow \quad sX + 5 + 2X + 6\frac{Y}{s} = -\frac{2}{s}$$

$$[sX - x(0)] + [sY - \ y(0)] + Y = 0 \quad \Rightarrow \quad sX + 5 + sY - 6 + Y = 0$$

$$s^2X + 5s + 2sX + 6Y = -2 \quad \Rightarrow \quad s(s+2)X + 6Y = -2 - 5s$$

$$(s+1)Y + sX = 1$$

$$s(s+2)(s+1)X + 6(s+1)Y = -(5s+2)(s+1)$$

$$-6(s+1)Y - 6sX = -6$$

Sumando las dos ecuaciones tenemos:

$$s(s+2)(s+1)X - 6sX = -(5s+2)(s+1) - 6$$

$$Xs(s^2 + 3s + 2 - 6) = -(5s+2)(s+1) - 6$$

$$X = -\frac{5s^2 + 7s + 2}{s(s+4)(s-1)} - \frac{6}{s(s+4)(s-1)}$$

Descomponiendo en fracciones parciales:

$$\frac{5s^2 + 7s + 2}{s(s+4)(s-1)} = \frac{A}{s} + \frac{B}{s+4} + \frac{C}{s-1}$$

$$5s^2 + 7s + 2 = A(s+4)(s-1) + Bs(s-1) + Cs(s+4)$$

$$5s^2 + 7s + 2 = As^2 + 3As - 4A + Bs^2 - Bs + Cs^2 + 4Cs$$

$$5s^2 + 7s + 2 = (A + B + C)s^2 + (3A - B + 4C)s - 4A$$

$5 = A + B + C$

$7 = 3A - B + 4C$

$2 = -4A$

Resolviendo el sistema A = -1/2 , B = 27/10 , C = 14/5

Ahora:

$$\frac{6}{s(s+4)(s-1)} = \frac{A'}{s} + \frac{B'}{s+4} + \frac{C'}{s-1}$$

$6 = (A' + B' + C')s^2 + (3A' - B' + 4C')s - 4A'$

$0 = A' + B' + C'$

$0 = 3A' - B' + 4C'$

$6 = -4A'$

Resolviendo el sistema A' = -3/2 , B' = 3/10 , C' = 6/5

Entonces:

$$\mathcal{L}^{-1}\{X\} = -\mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{B}{s+4} + \frac{C}{s-1}\right\} - \mathcal{L}^{-1}\left\{\frac{A'}{s} + \frac{B'}{s+4} + \frac{C'}{s-1}\right\}$$

$$x(t) = -(A + Be^{-4t} + Ce^{t}) - (A' + B'e^{-4t} + C'e^{t})$$

$$x(t) = -\left(-\frac{1}{2} + \frac{27}{10}e^{-4t} + \frac{14}{5}e^{t}\right) - \left(-\frac{3}{2} + \frac{3}{10}e^{-4t} + \frac{6}{5}e^{t}\right)$$

$$x(t) = -2 - 3e^{-4t} - 4e^{t}$$

Encontrando la segunda solución:

$(s+1)Y + sX = 1$

$$Y = \frac{1 - sX}{(s+1)}$$

$$Y = \frac{1}{s+1} - \frac{s}{(s+1)}\left[-\frac{5s^2 + 7s + 2}{s(s+4)(s-1)} - \frac{6}{s(s+4)(s-1)}\right]$$

$$Y = \frac{1}{s+1} + \frac{5s^2 + 7s + 2}{(s+4)(s-1)(s+1)} + \frac{6}{(s+4)(s-1)(s+1)}$$

Descomponiendo en fracciones parciales:

$$\frac{5s^2+7s+2}{(s+4)(s-1)(s+1)}=\frac{A}{s+4}+\frac{B}{s-1}+\frac{C}{s+1}$$

$$5s^2+7s+2=A(s-1)(s+1)+B(s+4)(s+1)+C(s+4)(s-1)$$

$$5s^2+7s+2=As^2-A+Bs^2+5Bs+4B+Cs^2+3Cs-4C$$

$$5s^2+7s+2=(A+B+C)s^2+(5B+3C)s+(4B-A-4C)$$

$$5=A+B+C$$

$$7=5B+3C$$

$$2=4B-A-4C$$

Resolviendo el sistema A = 18/5 , B = 7/5 , C = 0

$$\frac{6}{(s+4)(s-1)(s+1)}=\frac{A'}{s+4}+\frac{B'}{s-1}+\frac{C'}{s+1}$$

$$6=(A'+B'+C')s^2+(5B'+3C')s+(4B'-A'-4C')$$

$$0=A'+B'+C'$$

$$0=5B'+3C'$$

$$6=4B'-A'-4C'$$

Resolviendo el sistema A' = 6/15 , B' = 3/5 , C' = -1

Entonces:

$$\mathcal{L}^{-1}\{Y\}=\mathcal{L}^{-1}\left\{\frac{1}{s+1}+\frac{A}{s+4}+\frac{B}{s-1}+\frac{C}{s+1}+\frac{A'}{s+4}+\frac{B'}{s-1}+\frac{C'}{s+1}\right\}$$

$$y(t)=e^{-t}+Ae^{-4t}+Be^{t}+Ce^{-t}+A^{'}e^{-4t}+B^{'}e^{t}+C^{'}e^{-t}$$

$$y(t)=e^{-t}+\frac{18}{5}e^{-4t}+\frac{7}{5}e^{t}+\frac{6}{5}e^{-4t}+\frac{3}{5}e^{t}-e^{-t}$$

Finalmente:

$$\boxed{\boldsymbol{x(t)=-2-3e^{-4t}-4e^{t}}}$$

$$\boxed{\boldsymbol{y(t)=e^{-t}+\frac{24}{5}e^{-4t}+2e^{t}+\frac{3}{5}e^{t}-e^{-t}}}$$

$$\mathbf{2)}\ \begin{cases} \boldsymbol{x}' - \boldsymbol{y} = \begin{cases} \mathbf{0}\,, \mathbf{0} < t < 2 \\ \mathbf{1}\,, \mathbf{2} < t < 3 \\ \mathbf{0}\,, \qquad \boldsymbol{t} \geq \mathbf{3} \end{cases} \\ \boldsymbol{y}' - \boldsymbol{x} = \mathbf{1} \\ \quad \boldsymbol{x}(\mathbf{1}) = \boldsymbol{y}(\mathbf{1}) = \mathbf{1} \end{cases}$$

$$x' - y = u(t-2) - u(t-3)$$
$$y' - x = 1$$

$$\mathcal{L}\{x'\} - \mathcal{L}\{y\} = \mathcal{L}\{u(t-2)\} - \mathcal{L}\{u(t-3)\}$$
$$\mathcal{L}\{y'\} - \mathcal{L}\{x\} = \mathcal{L}\{1\}$$

$$[sX - x(0)] - Y = e^{-2s} - e^{-3s}$$
$$[sY - y(0)] - X = \frac{1}{s}$$

No conocemos el valor de x(0) y de y(0), pero vamos a llamar x(0) = w y y(0) = z , entonces:

$$sX - w - Y = e^{-2s} - e^{-3s}$$

$$s^2Y - zs - Xs = 1$$

Entonces:

$$X = \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} + \frac{w}{s} + \frac{Y}{s}$$

$$Y = \frac{1}{s^2} + \frac{z}{s} + \frac{X}{s}$$

Reemplazando Y

$$X = \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} + \frac{w}{s} + \frac{1}{s}\left(\frac{1}{s^2} + \frac{z}{s} + \frac{X}{s}\right)$$

$$X = \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} + \frac{w}{s} + \frac{1}{s^3} + \frac{z}{s^2} + \frac{X}{s^2}$$

$$X\left(1 - \frac{1}{s^2}\right) = \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} + \frac{w}{s} + \frac{1}{s^3} + \frac{z}{s^2}$$

$$X = \frac{s}{s^2-1}e^{-2s} - \frac{s}{s^2-1}e^{-3s} + w\frac{s}{s^2-1} + \frac{1}{s(s^2-1)} + \frac{z}{s^2-1}$$

$$\mathcal{L}^{-1}\{X\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2-1}e^{-2s} - \frac{s}{s^2-1}e^{-3s} + w\frac{s}{s^2-1} + \frac{1}{s(s^2-1)} + \frac{z}{s^2-1}\right\}$$

Resolviendo cada transformada inversa:

$$*\,\mathcal{L}^{-1}\left\{\frac{s}{s^2-1}e^{-2s}\right\} = u(t-2)\ Cosh(t-2)$$

$$*\,\mathcal{L}^{-1}\left\{\frac{s}{s^2-1}e^{-3s}\right\} = u(t-3)\ Cosh(t-3)$$

$$*\,\mathcal{L}^{-1}\left\{\frac{s}{s^2-1}\right\} = Cosh(t)$$

$$*\,\mathcal{L}^{-1}\left\{\frac{1}{s(s^2-1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} * \frac{1}{s^2-1}\right\}$$

$$\int_0^t Senh(x)dx \quad ; \quad Pero\ Senh\ x = \frac{e^x - e^{-x}}{2}$$

$$\int_0^t \frac{e^x - e^{-x}}{2}dx = \left[\frac{1}{2}(e^x + e^{-x})\right]_0^t \quad \Rightarrow \quad \int_0^t \frac{e^x - e^{-x}}{2}dx = \frac{1}{2}(e^t + e^{-t} - 2)$$

$$\int_0^t \frac{e^x - e^{-x}}{2}dx = \frac{e^t + e^{-t}}{2} - 1$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2-1)}\right\} = \frac{e^t + e^{-t}}{2} - 1$$

$$*\,\mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} = Senh(t)$$

Entonces:

$$x(t) = u(t-2)\ Cosh(t-2) - u(t-3)\ Cosh(t-3) + w\ Cosh(t) + \frac{e^t + e^{-t}}{2} - 1 + z\ Senh(t)$$

Ahora:

$$Y = \frac{1}{s^2} + \frac{z}{s} + \frac{1}{s}\left(\frac{s}{s^2-1}e^{-2s} - \frac{s}{s^2-1}e^{-3s} + w\frac{s}{s^2-1} + \frac{1}{s(s^2-1)} + \frac{z}{s^2-1}\right)$$

$$Y = \frac{1}{s^2} + \frac{z}{s} + \frac{1}{s^2-1}e^{-2s} - \frac{1}{s^2-1}e^{-3s} + w\frac{1}{s^2-1} + \frac{1}{s^2(s^2-1)} + \frac{z}{s(s^2-1)}$$

Resolviendo cada transformada inversa:

$$*\,\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$$

$$*\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$

$$*\,\mathcal{L}^{-1}\left\{\frac{1}{s^2-1}e^{-2s}\right\} = u(t-2)\ Senh(t-2)$$

$$*\,\mathcal{L}^{-1}\left\{\frac{1}{s^2-1}e^{-3s}\right\} = u(t-3)\ Senh(t-3)$$

$$*\,\mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} = Senh(t)$$

$$*\,\mathcal{L}^{-1}\left\{\frac{1}{s(s^2-1)}\right\} = \frac{e^t+e^{-t}}{2} - 1$$

$$*\,\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2-1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2} * \frac{1}{s^2-1}\right\}$$

$$\int_0^t (t-x)Senh(x)dx = \int_0^t (t-x)\left(\frac{e^x-e^{-x}}{2}\right)dx$$

$$t\int_0^t \frac{e^x-e^{-x}}{2}dx - \frac{1}{2}\left(\int_0^t x\,e^x dx - \int_0^t x\,e^{-x}dx\right)$$

$$\left[\frac{t}{2}(e^x+e^{-x}) - \frac{1}{2}[e^x(x-1)+e^{-x}(x+1)]\right]_0^t$$

Evaluando:

$$\frac{t}{2}(e^t+e^{-t}) - \frac{1}{2}[e^t(t-1)+e^{-t}(t+1)] - t = \frac{te^t}{2} + \frac{te^{-t}}{2} - \frac{te^t}{2} + \frac{e^t}{2} - \frac{te^{-t}}{2} - \frac{e^{-t}}{2} - t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2-1)}\right\} = \frac{e^t}{2} - \frac{e^{-t}}{2} - t$$

Por lo tanto:

$$x(t) = u(t-2)\,Cosh(t-2) - u(t-3)\,Cosh(t-3) + w\,Cosh(t) + \frac{e^t+e^{-t}}{2} - 1 + z\,Senh(t)$$

$$y(t) = t + z + u(t-2)\,Senh(t-2) - u(t-3)\,Senh(t-3) + w\,Senh(t) + \frac{e^t}{2} - \frac{e^{-t}}{2} - t + z\left(\frac{e^t+e^{-t}}{2} - 1\right)$$

Encontrando los valores de "w" y "z"

Sabemos que x(0) = w y y(0) = z

$$x(0) = u(0-2)\,Cosh(0-2) - u(0-3)\,Cosh(0-3) + w\,Cosh(0) + \frac{e^0+e^{-0}}{2} - 1 + z\,Senh(0)$$

$$w = u(-2)\,Cosh(-2) - u(-3)\,Cosh(-3) + w\left(\frac{e^0+e^{-0}}{2}\right) + \frac{e^0+e^{-0}}{2} - 1 + z\left(\frac{e^0-e^{-0}}{2}\right)$$

$$w = 0 - 0 + \frac{w}{2} + \frac{1}{2} - 1 + \frac{z}{2}$$

$$\frac{3w}{2} = \frac{z}{2} - \frac{1}{2}$$

$$w = \frac{1}{3}(z-1)$$

$$y(0) = 0 + z + u(0-2)\, Senh(0-2) - u(0-3)\, Senh(0-3) + w\, Senh(0) + \frac{e^0}{2} - \frac{e^{-0}}{2} - 0 + z\left(\frac{e^0 + e^{-0}}{2} - 1\right)$$

$$z = z + u(-2)\, Senh(-2) - u(-3)\, Senh(-3) + w\, \left(\frac{e^0 - e^{-0}}{2}\right) + \frac{e^0}{2} - \frac{e^{-0}}{2} + z\left(\frac{e^0 + e^{-0}}{2} - 1\right)$$

$$z = z + 0 - 0 + \frac{w}{2} + \frac{z}{2} - z$$

$$z = \frac{w}{2} + \frac{z}{2}$$

$$\frac{z}{2} = \frac{w}{2} \quad \Rightarrow \quad z = w$$

Reemplazando nos queda:

$$w = \frac{1}{3}(w-1) \quad \Rightarrow \quad \frac{2w}{3} = -\frac{1}{3} \quad \Rightarrow \quad w = -\frac{1}{2}$$

$$z = -\frac{1}{2}$$

Finalmente:

$$x(t) = u(t-2)\, Cosh(t-2) - u(t-3)\, Cosh(t-3) - \frac{1}{2}\, Cosh(t) + \frac{e^t + e^{-t}}{2} - 1 - \frac{1}{2}\, Senh(t)$$

$$y(t) = t - \frac{1}{2} + u(t-2)\, Senh(t-2) - u(t-3)\, Senh(t-3) - \frac{1}{2}\, Senh(t) + \frac{e^t}{2} - \frac{e^{-t}}{2} + \frac{1}{2} - \frac{1}{2}\left(\frac{e^t + e^{-t}}{2} - 1\right)$$

$$x(t) = u(t-2)\, Cosh(t-2) - u(t-3)\, Cosh(t-3) - \frac{1}{2}\, Cosh(t) + \, Cosh(t) - 1 - \frac{1}{2}\, Senh(t)$$

$$y(t) = t - \frac{1}{2} + u(t-2)\, Senh(t-2) - u(t-3)\, Senh(t-3) - \frac{1}{2}\, Senh(t) + Senh(t) + \frac{1}{2} - \frac{1}{2}(Cosh(t) - 1)$$

$$\boxed{\boldsymbol{x(t) = u(t-2)\, Cosh(t-2) - u(t-3)\, Cosh(t-3) + \frac{1}{2}\, Cosh(t) - 1 - \frac{1}{2}\, Senh(t)}}$$

$$\boxed{\boldsymbol{y(t) = t + u(t-2)\, Senh(t-2) - u(t-3)\, Senh(t-3) + \frac{1}{2}\, Senh(t) - \frac{1}{2}(Cosh(t) - 1)}}$$

$$3)\begin{cases} \boldsymbol{x' - y' = Sen(t)\, u(t-\pi)} \\ \boldsymbol{x + y' = 0} \\ \boldsymbol{x(0) = y(0) = 1} \end{cases}$$

$x' - y' = Sen[(t-\pi)+\pi]\ u(t-\pi)$
$x + y' = 0$

$x' - y' = [Sen(t-\pi)Cos\pi + Cos(t-\pi)Sen\pi]\ u(t-\pi)$
$x + y' = 0$

$x' - y' = -Sen(t-\pi)\ u(t-\pi)$
$x + y' = 0$

$\mathcal{L}\{x'\} - \mathcal{L}\{y'\} = -\mathcal{L}\{Sen(t-\pi)\ u(t-\pi)\}$
$\mathcal{L}\{x\} + \mathcal{L}\{y'\} = 0$

$sX - x(0) - sY + y(0) = -e^{-\pi s}\dfrac{1}{s^2+1}$
$X + sY - y(0) = 0$

$sX - sY = -e^{-\pi s}\dfrac{1}{s^2+1}$
$X + sY = 1$

Usando la regla de Kramer tenemos:

$$X = \frac{\begin{vmatrix} -e^{-\pi s}\dfrac{1}{s^2+1} & -s \\ 1 & s \end{vmatrix}}{\begin{vmatrix} s & -s \\ 1 & s \end{vmatrix}} \quad ; \quad Y = \frac{\begin{vmatrix} s & -e^{-\pi s}\dfrac{1}{s^2+1} \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} s & -s \\ 1 & s \end{vmatrix}}$$

$$X = \frac{-se^{-\pi s}\dfrac{1}{s^2+1} + s}{s^2+s} = \frac{1}{s+1} - \frac{1}{(s^2+1)(s+1)}e^{-\pi s}$$

$$Y = \frac{s - e^{-\pi s}\dfrac{1}{s^2+1}}{s^2+s} = \frac{1}{s+1} - \frac{1}{s(s^2+1)(s+1)}e^{-\pi s}$$

<u>Encontrando la 1era solución</u>

Aplicando transformada inversa:

$$\mathcal{L}^{-1}\{X\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1} - \frac{1}{(s^2+1)(s+1)}e^{-\pi s}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = te^{-t}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s+1)}e^{-\pi s}\right\}\ ;\ Aplicando\ convolución\ tenemos\ \mathcal{L}^{-1}\left\{\underbrace{\frac{1}{s^2+1} * \frac{1}{s+1}}_{Sen(t)\, *e^{-t}}\right\}$$

$$\int_0^t e^{-(t-x)}Sen(x)dx = e^{-t}\int_0^t e^x Sen(x)dx$$

Resolviendo la integral por partes tenemos:

$$\int e^x Sen(x)dx = -e^x Cos(x) + \int e^x Cos(x)dx$$

$$\int e^x Sen(x)dx = -e^x Cos(x) + \left[e^x Sen(x) - \int e^x Sen(x)dx\right]$$

$$\int e^x Sen(x)dx = \frac{1}{2}[e^x Sen(x) - e^x Cos(x)]$$

Evaluando:

$$\left[\frac{1}{2}[e^x Sen(x) - e^x Cos(x)]\right]_0^t = \frac{1}{2}[e^t Sen(t) - e^t Cos(t) + 1]$$

Entonces:

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s+1)}\right\} = \frac{e^{-t}}{2}[e^t Sen(t) - e^t Cos(t) + 1] = \frac{1}{2}[Sen(t) - Cos(t) + e^{-t}]$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s+1)}e^{-\pi s}\right\} = \frac{1}{2}\left[Sen(t-\pi) - Cos(t-\pi) + e^{-(t-\pi)}\right]$$

Luego:

$$x(t) = te^{-t} - \frac{1}{2}\left[Sen(t-\pi) - Cos(t-\pi) + e^{-(t-\pi)}\right]$$

Encontrando la 2da solución

Aplicando transformada inversa:

$$\mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1} - \frac{1}{s(s^2+1)(s+1)}e^{-\pi s}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = te^{-t}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)(s+1)}e^{-\pi s}\right\} \quad ; \quad \textit{Aplicando convolución tenemos } \mathcal{L}^{-1}\left\{\frac{1}{s} * \frac{1}{s^2+1} * \frac{1}{s+1}\right\}$$

$$\mathcal{L}^{-1}\left\{\underbrace{\frac{1}{s} * \left(\frac{1}{s^2+1} * \frac{1}{s+1}\right)}_{1 * \frac{1}{2}[Sen(t) - Cos(t) + e^{-t}]}\right\}$$

$$\int_0^t \frac{1}{2}[Sen(x) - Cos(x) + e^{-x}]dx$$

$$\frac{1}{2}\int_0^t [Sen(x) - Cos(x) + e^{-x}]dx = \left[\frac{1}{2}[-Cos(x) - Sen(x) - e^{-x}]\right]_0^t$$

$$\frac{1}{2}\int_0^t [Sen(x) - Cos(x) + e^{-x}]dx = \frac{1}{2}[-Cos(t) - Sen(t) - e^{-t} + 2]$$

Entonces:

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)(s+1)}e^{-\pi s}\right\} = \frac{1}{2}\left[-Cos(t-\pi) - Sen(t-\pi) - e^{-(t-\pi)} + 2\right]$$

Luego:

$$y(t) = te^{-t} + \frac{1}{2}\left[Cos(t-\pi) + Sen(t-\pi) + e^{-(t-\pi)} - 2\right]$$

Finalmente la solución del sistema es:

$$\boxed{\boldsymbol{x(t) = te^{-t} - \frac{1}{2}\left[Sen(t-\pi) - Cos(t-\pi) + e^{-(t-\pi)}\right]}}$$

$$\boxed{\boldsymbol{y(t) = te^{-t} + \frac{1}{2}\left[Cos(t-\pi) + Sen(t-\pi) + e^{-(t-\pi)} - 2\right]}}$$

APLICACIONES

SISTEMA MASA – RESORTE - AMORTIGUADOR

1) Una masa de 1 kg está unida a un resorte ligero que es estirado 2m por una fuerza de 8 N, la masa se encuentra inicialmente en reposo en su posición de equilibrio. Iniciando en el tiempo t = 0 seg se le aplica una fuerza externa f(t)=Cos(2t) a la masa pero en el instante t = 2π esta cesa abruptamente y la masa queda libre continuando con su movimiento, pero en el tiempo t = 4π, la masa es golpeada hacia abajo con un martillo con una fuerza de 10N. Determine la ecuación del movimiento, además la posición de la masa cuando t = 9π/4 seg.

$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=f(t)$$

Nos dice que el resorte es estirado 2mpor una fuerza de 8N, entonces:

$$F=kx \quad \Rightarrow \quad k=\frac{F}{x}=\frac{8}{2} \quad \Rightarrow \quad k=4\,N/m$$

Además nos dice, que en t=0 se le aplica una fuerza externa, y después cesa abruptamente, entonces f(t) nos queda:

$$f(t)=\begin{cases} Cos(2t)\,; & 0\le t<2\pi \\ 0 & ;\; t>2\pi \end{cases}$$

Pero en t = 4 π, es golpeado con un martillo, produciendo un impulso, entonces, nuestra ecuación nos queda:

$$\frac{d^2x}{dt^2}+4x=(u_0-u_{2\pi})Cos(2t)+10\,\delta(t-4\pi)$$

$$x''+4x=u_0Cos(2t)-u_{2\pi}Cos(2t)+10\delta(t-4\pi)$$

La función coseno ya está desfasada, entonces aplicando transformada de Laplace, nos queda:

$$[s^2X-s\,x(0)-x'(0)]+4X=\frac{s}{s^2+4}-\frac{s}{s^2+4}e^{-2\pi s}+10e^{-4\pi s}$$

Sabemos que en t = 0 , x(0) = x'(0) = 0

$$s^2X+4X=\frac{s}{s^2+4}-\frac{s}{s^2+4}e^{-2\pi s}+10e^{-4\pi s}$$

$$X(s^2+4)=\frac{s}{s^2+4}-\frac{s}{s^2+4}e^{-2\pi s}+10e^{-4\pi s}$$

$$X=\frac{s}{(s^2+4)^2}-\frac{s}{(s^2+4)^2}e^{-2\pi s}+10\frac{e^{-4\pi s}}{s^2+4}$$

Aplicando transformada inversa:

$$\mathcal{L}^{-1}\{X\}=\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)^2}\right\}-\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)^2}e^{-2\pi s}\right\}+10\mathcal{L}^{-1}\left\{\frac{e^{-4\pi s}}{s^2+4}\right\}$$

Aplicando convolución:

$$\mathcal{L}^{-1}\left\{\underbrace{\frac{s}{s^2+4}}_{Cos(2t)} * \overbrace{\frac{1}{s^2+4}}^{\frac{1}{2}Sen(2t)}\right\}$$

$$\frac{1}{2}\int_0^t Cos(2x)Sen[2(t-x)]dx$$

$$\frac{1}{4}\int_0^t [Sen(2x+2t-2x) - Sen(2x-2t+2x)]dx$$

$$\frac{1}{4}\int_0^t [Sen(2t) - Sen(4x-2t)]dx$$

$$\frac{1}{4}\left[x\ Sen(2t) + \frac{1}{4}Cos(4x-2t)\right]_0^t \quad \Rightarrow \quad \frac{1}{4}\left[t\ Sen(2t) + \frac{1}{4}Cos(2t) - \frac{1}{4}Cos(-2t)\right]$$

Sabemos que Cos(-x) = Cos(x), entonces:

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} = \frac{1}{4}t\ Sen(2t)$$

Finalmente:

$$\boxed{\boldsymbol{x(t) = \frac{1}{4}t\ Sen(2t) - \frac{1}{4}u(t-2\pi)(t-2\pi)\ Sen[2(t-2\pi)] + 5\ u(t-4\pi)\ Sen[2(t-4\pi)]}}$$

<u>Encontrando la posición de la masa en t = 9π/4 seg</u>

$$x\left(\frac{9\pi}{4}\right) = \frac{1}{4}\left(\frac{9\pi}{4}\right)\ Sen\left[2\left(\frac{9\pi}{4}\right)\right] - \frac{1}{4}u\left[\left(\frac{9\pi}{4}\right) - 2\pi\right]\left(\frac{9\pi}{4} - 2\pi\right)\ Sen\left[2\left(\frac{9\pi}{4} - 2\pi\right)\right] + 5\ u\left(\frac{9\pi}{4} - 4\pi\right)\ Sen\left[2\left(\frac{9\pi}{4} - 4\pi\right)\right]$$

$$x\left(\frac{9\pi}{4}\right) = \frac{9\pi}{16}\ Sen\left(\frac{9\pi}{2}\right) - \frac{\pi}{16}u\left(\frac{\pi}{4}\right)Sen\left(\frac{\pi}{2}\right) + 5\ u\left(-\frac{7\pi}{4}\right)Sen\left(-\frac{7\pi}{2}\right)$$

$$x\left(\frac{9\pi}{4}\right) = \frac{9\pi}{16}(1) - \frac{\pi}{16}(1)(1) + 5(0)(1)$$

$$x\left(\frac{9\pi}{4}\right) = \frac{9\pi}{16} - \frac{\pi}{16}$$

$$\boxed{\boldsymbol{x\left(\frac{9\pi}{4}\right) = \frac{\pi}{2}[m]}}$$

2) En el extremo de un resorte espiral que está sujeto al techo se coloca un cuerpo de masa igual a 1 kg. El resorte se ha alargado 2m hasta quedar en reposo en su posición de equilibrio. En t = 0 el cuerpo es desplazado 50 cm por debajo de la posición de equilibrio y lanzado con una velocidad inicial de 1m/seg dirigida hacia arriba. El sistema consta también de un amortiguador cuyo coeficiente de amortiguamiento es de 2.5 N.seg/m. Desde t = 0, una fuerza externa es aplicada al cuerpo, la misma que está dada por f(t) = Sen$(\pi t/2)$. En t = 10 seg y en t = 20 seg el cuerpo es golpeado hacia abajo proporcionando una fuerza de 5N y de 10N, respectivamente. (use g = 10 m/seg^2). Determine la ecuación del movimiento

$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=f(t)$$

Nos dice que el resorte se ha alargado 2m hasta quedar en reposo al colocar una masa de 1 kg, entonces:

$$F=kx \quad\Rightarrow\quad k=\frac{mg}{x}=\frac{1(10)}{2} \quad\Rightarrow\quad k=5\ N/m$$

Además nos dice que en t=10 y en t=20 el cuerpo es golpeado hacia abajo, es decir recibe un impulso, entonces nuestra ecuación es la siguiente:

$$(1)\frac{d^2x}{dt^2}+(2.5)\frac{dx}{dt}+(5)x=Sen\left(\frac{\pi}{2}t\right)+5\,\delta(t-10)+10\,\delta(t-20)$$

$$x''+2.5x'+5x=Sen\left(\frac{\pi}{2}t\right)+5\,\delta(t-10)+10\,\delta(t-20)$$

Aplicando transformada de Laplace:

$$\mathcal{L}\{x''\}+2.5\,\mathcal{L}\{x'\}+5\,\mathcal{L}\{x\}=\mathcal{L}\left\{Sen\left(\frac{\pi}{2}t\right)\right\}+5\,\{\delta(t-10)\}+10\,\{\delta(t-20)\}$$

$$[s^2X-s\,x(0)-x'(0)]+2.5[sX-x(0)]+5X=\frac{\frac{\pi}{2}}{s^2+\frac{\pi^2}{4}}+5e^{-10s}+10e^{-20s}$$

Sabemos que en t = 0 el cuerpo es lanzado con una velocidad inicial hacia arriba y además es desplazado 50 cm por debajo de su posición de equilibrio, entonces:

$$[s^2X-0.5s+1]+2.5[sX-0.5]+5X=\frac{\frac{\pi}{2}}{s^2+\frac{\pi^2}{4}}+5e^{-10s}+10e^{-20s}$$

$$s^2X-0.5\,s+1+2.5\,sX-1.25+5X=\frac{\frac{\pi}{2}}{s^2+\frac{\pi^2}{4}}+5e^{-10s}+10e^{-20s}$$

$$X\left(s^2+\frac{5}{2}s+5\right)=\frac{\frac{\pi}{2}}{s^2+\frac{\pi^2}{4}}+5e^{-10s}+10e^{-20s}+\frac{1}{4}$$

$$X=\frac{\frac{\pi}{2}}{\left(s^2+\frac{\pi^2}{4}\right)\left[\left(s+\frac{5}{4}\right)^2+\frac{15}{8}\right]}+5\frac{e^{-10s}}{\left(s+\frac{5}{4}\right)^2+\frac{15}{8}}+10\frac{e^{-20s}}{\left(s+\frac{5}{4}\right)^2+\frac{15}{8}}+\frac{1}{4}\frac{1}{\left(s+\frac{5}{4}\right)^2+\frac{15}{8}}$$

Aplicando la transformada inversa:

$$\mathcal{L}^{-1}\{X\}=\mathcal{L}^{-1}\left\{\frac{\frac{\pi}{2}}{\left(s^2+\frac{\pi^2}{4}\right)\left[\left(s+\frac{5}{4}\right)^2+\frac{15}{8}\right]}\right\}+5\mathcal{L}^{-1}\left\{\frac{e^{-10s}}{\left(s+\frac{5}{4}\right)^2+\frac{15}{8}}\right\}+10\mathcal{L}^{-1}\left\{\frac{e^{-20s}}{\left(s+\frac{5}{4}\right)^2+\frac{15}{8}}\right\}+\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{\left(s+\frac{5}{4}\right)^2+\frac{15}{8}}\right\}$$

Aplicando convolución:

$$\mathcal{L}^{-1}\left\{\frac{\frac{\pi}{2}}{s^2+\frac{\pi^2}{4}}*\left(\sqrt{\frac{8}{15}}\right)\frac{\sqrt{\frac{15}{8}}}{\left(s+\frac{5}{4}\right)^2+\frac{15}{8}}\right\}$$

$$\left(\sqrt{\frac{8}{15}}\right)\mathcal{L}^{-1}\underbrace{\left\{\frac{\frac{\pi}{2}}{s^2+\frac{\pi^2}{4}}*\frac{\sqrt{\frac{15}{8}}}{\left(s+\frac{5}{4}\right)^2+\frac{15}{8}}\right\}}_{Sen\left(\frac{\pi}{2}t\right)*e^{-\frac{5}{4}t}\,Sen\left(\sqrt{\frac{15}{8}}t\right)}$$

$$\int_0^t Sen\left[\frac{\pi}{2}(t-x)\right]e^{-\frac{5}{4}x}\,Sen\left(\sqrt{\frac{15}{8}}x\right)dx$$

Resolviendo la integral y para mayor comodidad, tenemos:

$$\int Sen[A(t-x)]e^{Bx}\,Sen(Cx)dx$$

$$\int e^{Bx}Sen(At-Ax)\,Sen(Cx)dx$$

Sabemos que:

$$Sen(mx)\,Sen(nx)=\frac{1}{2}[Cos(m-n)x-Cos(m+n)x]$$

Entonces:

$$\int e^{Bx}\frac{1}{2}[Cos(At-Ax-Cx)-Cos(At-Ax+Cx)]dx$$

$$\frac{1}{2}\left[\int e^{Bx}Cos[At-(A+C)x]dx-\int e^{Bx}Cos[At+(C-A)x]dx\right]$$

Resolviendo la integral por partes, tenemos:

$$u=e^{Bx}\quad\Rightarrow\quad du=Be^{Bx}dx$$

$$dv=Cos[At-(A+C)x]dx\quad\Rightarrow\quad v=-\frac{1}{A+C}Sen[At-(A+C)x]$$

$$\int e^{Bx}Cos[At-(A+C)x]dx=-\frac{e^{Bx}}{A+C}Sen[At-(A+C)x]+\frac{B}{A+C}\int e^{Bx}Sen[At-(A+C)x]dx$$

Integrando nuevamente por partes:

$$u = e^{Bx} \quad \Rightarrow \quad du = Be^{Bx}dx$$

$$dv = Sen[At-(A+C)x]dx \quad \Rightarrow \quad v = \frac{1}{A+C}Cos[At-(A+C)x]$$

$$\int e^{Bx}Cos[At-(A+C)x]dx = -\frac{e^{Bx}}{A+C}Sen[At-(A+C)x] + \frac{B}{A+C}\left[\frac{e^{Bx}}{A+C}Cos[At-(A+C)x] - \frac{B}{A+C}\int e^{Bx}Cos[At-(A+C)x]dx\right]$$

$$\int e^{Bx}Cos[At-(A+C)x]dx = -\frac{e^{Bx}}{A+C}Sen[At-(A+C)x] + \frac{Be^{Bx}}{(A+C)^2}Cos[At-(A+C)x] - \left(\frac{B}{A+C}\right)^2\int e^{Bx}Cos[At-(A+C)x]dx$$

$$\int e^{Bx}Cos[At-(A+C)x]dx = \left[1+\left(\frac{B}{A+C}\right)^2\right]^{-1}\left[\frac{Be^{Bx}}{(A+C)^2}Cos[At-(A+C)x] - \frac{e^{Bx}}{A+C}Sen[At-(A+C)x]\right]$$

Luego tenemos que:

$$\left[1+\left(\frac{B}{A+C}\right)^2\right]^{-1}\left[\frac{Be^{Bx}}{(A+C)^2}Cos[At-(A+C)x] - \frac{e^{Bx}}{A+C}Sen[At-(A+C)x]\right]_0^t$$

$$\left[1+\left(\frac{B}{A+C}\right)^2\right]^{-1}\left[\begin{array}{l}\frac{Be^{Bt}}{(A+C)^2}Cos[At-(A+C)t] - \frac{e^{Bt}}{A+C}Sen[At-(A+C)t] \\ -\frac{Be^{B(0)}}{(A+C)^2}Cos[At-(A+C)(0)] + \frac{e^{B(0)}}{A+C}Sen[At-(A+C)(0)]\end{array}\right]$$

$$\left[1+\left(\frac{B}{A+C}\right)^2\right]^{-1}\left[\frac{Be^{Bt}}{(A+C)^2}Cos(-Ct) - \frac{e^{Bt}}{A+C}Sen(-Ct) - \frac{B}{(A+C)^2}Cos(At) + \frac{1}{A+C}Sen(At)\right]$$

Resolviendo la segunda integral:

$$\int e^{Bx}Cos[At+(A+C)x]dx = \left[1+\left(\frac{B}{A+C}\right)^2\right]^{-1}\left[\frac{Be^{Bx}}{(A+C)^2}Cos[At+(A+C)x] - \frac{e^{Bx}}{A+C}Sen[At+(A+C)x]\right]$$

Luego tenemos que:

$$\left[1+\left(\frac{B}{A+C}\right)^2\right]^{-1}\left[\frac{Be^{Bx}}{(A+C)^2}Cos[At+(A+C)x] - \frac{e^{Bx}}{A+C}Sen[At+(A+C)x]\right]_0^t$$

$$\left[1+\left(\frac{B}{A+C}\right)^2\right]^{-1}\left[\begin{array}{l}\frac{Be^{Bt}}{(A+C)^2}Cos[At+(A+C)t] - \frac{e^{Bt}}{A+C}Sen[At+(A+C)t] \\ -\frac{Be^{B(0)}}{(A+C)^2}Cos[At+(A+C)(0)] + \frac{e^{B(0)}}{A+C}Sen[At+(A+C)(0)]\end{array}\right]$$

$$\left[1+\left(\frac{B}{A+C}\right)^2\right]^{-1}\left[\frac{Be^{Bt}}{(A+C)^2}Cos[(2A+C)t] - \frac{e^{Bt}}{A+C}Sen[(2A+C)t] - \frac{B}{(A+C)^2}Cos(At) + \frac{1}{A+C}Sen(At)\right]$$

Entonces, la transformada inversa nos queda:

$$\left(\sqrt{\frac{8}{15}}\right)\mathcal{L}^{-1}\left\{\frac{\frac{\pi}{2}}{s^2+\frac{\pi^2}{4}} * \frac{\sqrt{\frac{15}{8}}}{\left(s+\frac{5}{4}\right)^2+\frac{15}{8}}\right\}$$

$$\frac{1}{2}\left(\sqrt{\frac{8}{15}}\right)\left[1+\left(\frac{B}{A+C}\right)^2\right]^{-2}\left[\frac{Be^{Bt}}{(A+C)^2}Cos(Ct)+\frac{e^{Bt}}{A+C}Sen(-Ct)-\frac{B}{(A+C)^2}Cos(At)+\frac{1}{A+C}Sen(At)\right]$$
$$\left[\frac{Be^{Bt}}{(A+C)^2}Cos[(2A+C)t]-\frac{e^{Bt}}{A+C}Sen[(2A+C)t]-\frac{B}{(A+C)^2}Cos(At)+\frac{1}{A+C}Sen(At)\right]$$

$$\frac{1}{2}\left(\sqrt{\frac{8}{15}}\right)\left[1+\left(-\frac{5/4}{\frac{\pi}{2}+\sqrt{\frac{15}{8}}}\right)^2\right]^{-2}\left[-\frac{5e^{-\frac{5}{4}t}}{4\left(\frac{\pi}{2}+\sqrt{\frac{15}{8}}\right)^2}Cos\left(\sqrt{\frac{15}{8}}t\right)+\frac{e^{-\frac{5}{4}t}}{\frac{\pi}{2}+\sqrt{\frac{15}{8}}}Sen\left(-\sqrt{\frac{15}{8}}t\right)+\frac{5}{4\left(\frac{\pi}{2}+\sqrt{\frac{15}{8}}\right)^2}Cos\left(\frac{\pi}{2}t\right)+\frac{1}{\frac{\pi}{2}+\sqrt{\frac{15}{8}}}Sen\left(\frac{\pi}{2}t\right)\right]$$
$$\left[-\frac{5e^{-\frac{5}{4}t}}{4\left(\frac{\pi}{2}+\sqrt{\frac{15}{8}}\right)^2}Cos\left[\left(\pi+\sqrt{\frac{15}{8}}\right)t\right]-\frac{e^{-\frac{5}{4}t}}{\frac{\pi}{2}+\sqrt{\frac{15}{8}}}Sen\left[\left(\pi+\sqrt{\frac{15}{8}}\right)t\right]+\frac{5}{4\left(\frac{\pi}{2}+\sqrt{\frac{15}{8}}\right)^2}Cos\left(\frac{\pi}{2}t\right)+\frac{1}{\frac{\pi}{2}+\sqrt{\frac{15}{8}}}Sen\left(\frac{\pi}{2}t\right)\right]$$

Resolviendo las demás transformadas inversas:

$$\mathcal{L}^{-1}\left\{\frac{e^{-10s}}{\left(s+\frac{5}{4}\right)^2+\frac{15}{8}}\right\}=\sqrt{\frac{8}{15}}e^{-\frac{5}{4}(t-10)}\,Sen\left[\sqrt{\frac{15}{8}}(t-10)\right]u(t-10)$$

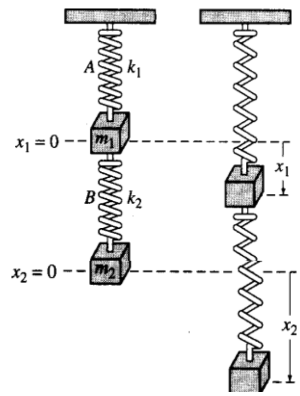
$$\mathcal{L}^{-1}\left\{\frac{e^{-20s}}{\left(s+\frac{5}{4}\right)^2+\frac{15}{8}}\right\}=\sqrt{\frac{8}{15}}e^{-\frac{5}{4}(t-20)}\,Sen\left[\sqrt{\frac{15}{8}}(t-20)\right]u(t-20)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{\left(s+\frac{5}{4}\right)^2+\frac{15}{8}}\right\}=\sqrt{\frac{8}{15}}e^{-\frac{5}{4}t}\,Sen\left(\sqrt{\frac{15}{8}}t\right)$$

Finalmente la ecuación del movimiento es:

$$\boldsymbol{x(t)=\frac{1}{2}\left(\sqrt{\frac{8}{15}}\right)\left[1+\left(-\frac{5/4}{\frac{\pi}{2}+\sqrt{\frac{15}{8}}}\right)^2\right]^{-2}\left[-\frac{5e^{-\frac{5}{4}t}}{4\left(\frac{\pi}{2}+\sqrt{\frac{15}{8}}\right)^2}Cos\left(\sqrt{\frac{15}{8}}t\right)+\frac{e^{-\frac{5}{4}t}}{\frac{\pi}{2}+\sqrt{\frac{15}{8}}}Sen\left(-\sqrt{\frac{15}{8}}t\right)+\frac{5}{4\left(\frac{\pi}{2}+\sqrt{\frac{15}{8}}\right)^2}Cos\left(\frac{\pi}{2}t\right)+\frac{1}{\frac{\pi}{2}+\sqrt{\frac{15}{8}}}Sen\left(\frac{\pi}{2}t\right)\right]}$$
$$\boldsymbol{\left[-\frac{5e^{-\frac{5}{4}t}}{4\left(\frac{\pi}{2}+\sqrt{\frac{15}{8}}\right)^2}Cos\left[\left(\pi+\sqrt{\frac{15}{8}}\right)t\right]-\frac{e^{-\frac{5}{4}t}}{\frac{\pi}{2}+\sqrt{\frac{15}{8}}}Sen\left[\left(\pi+\sqrt{\frac{15}{8}}\right)t\right]+\frac{5}{4\left(\frac{\pi}{2}+\sqrt{\frac{15}{8}}\right)^2}Cos\left(\frac{\pi}{2}t\right)+\frac{1}{\frac{\pi}{2}+\sqrt{\frac{15}{8}}}Sen\left(\frac{\pi}{2}t\right)\right]}$$
$$\boldsymbol{+5\sqrt{\frac{15}{8}}e^{-\frac{5}{4}(t-10)}\,Sen\left[\sqrt{\frac{15}{8}}(t-10)\right]u(t-10)+10\sqrt{\frac{15}{8}}e^{-\frac{5}{4}(t-20)}\,Sen\left[\sqrt{\frac{15}{8}}(t-20)\right]u(t-20)+\frac{1}{4}\sqrt{\frac{15}{8}}e^{-\frac{5}{4}t}\,Sen\left(\sqrt{\frac{15}{8}}t\right)}$$

3) Dos masas $m_1 = 1kg$ y $m_2 = 1kg$ estan unidas a dos resortes A y B, de masa insignificante cuyas constantes de resorte son $k_1 = 6\ N/m$ y $k_2 = 4N/m$ respectivamente, y los resortes se fijan como se ve en la figura. Además la masa m_1 parte con una velocidad de 1 m/seg hacia abajo y la masa m_2 parte con una velocidad de 1 m/seg hacia arriba, si no se aplica fuerzas externas al sistema, y en ausencia de fuerza de amortiguamiento encuentre los desplazamientos verticales de las masas respecto a sus posiciones de equilibrio.



Cuando el sistema está en movimiento, el resorte B es sometido a alargamiento y compresión, y en ausencia de fuerzas externas su alargamiento neto es $x_1 - x_2$ entonces según la ley de Hooke vemos que los resortes A y B ejercen fuerzas $-k_1x_1$ y $k_2(x_2 - x_1)$ respectivamente sobre m_1, entonces la fuerza neta sobre m_1 es $-k_1x_1 + k_2(x_2 - x_1)$, entonces según la 2da ley de Newton podemos escribir:

$$m_1\frac{d^2x_1}{dt^2} = -k_1x_1 + k_2(x_2 - x_1)$$

De igual forma, la fuerza neta ejercida sobre la masa m_2 sólo se debe al alargamiento neto de B; esto es, $-k_2(x_2 - x_1)$. En consecuencia:

$$m_2\frac{d^2x_1}{dt^2} = -k_2(x_2 - x_1)$$

En otras palabras, el movimiento del sistema acoplado se representa con el sistema de ecuaciones diferenciales simultáneas de 2do orden:

$$m_1x''_1 = -k_1x_1 + k_2(x_2 - x_1)$$

$$m_2x''_2 = -k_2(x_2 - x_1)$$

Entonces:

$m_1x''_1 + (k_1 + k_2)x_1 - k_2x_2 = 0$

$m_2x''_2 + k_2x_2 - k_2x_1 = 0$

Aplicando transformada de Laplace:

$\mathcal{L}\{x''_1\} + 10\mathcal{L}\{x_1\} - 4\mathcal{L}\{x_2\} = 0$

$\mathcal{L}\{x''_2\} + 4\mathcal{L}\{x_2\} - 4\mathcal{L}\{x_1\} = 0$

Entonces:

$[s^2X_1 - sx_1(0) - x'_1(0)] + 10X_1 - 4X_2 = 0$

$[s^2X_2 - sx_2(0) - x'_2(0)] + 4X_2 - 4X_1 = 0$

Tenemos bien claro que, $x_1(0) = x'_1(0) = 1$ *y* $x_2(0) = x'_2(0) = -1$ *entonces:*

$s^2X_1 - 1 + 10X_1 - 4X_2 = 0 \quad \Rightarrow \quad (s^2 + 10)X_1 - 4X_2 = 1$

$s^2X_2 + 1 + 4X_2 - 4X_1 = 0 \quad \Rightarrow \quad (s^2 + 4)X_2 - 4X_1 = -1$

Despejando X_2 *de ambas ecuaciones e igualando tenemos:*

$$(s^2 + 10)X_1 - 1 = \frac{16X_1 - 4}{(s^2 + 4)}$$

$(s^2 + 10)(s^2 + 4)X_1 - (s^2 + 4) = 16X_1 - 4$

$[(s^2 + 10)(s^2 + 4) - 16]X_1 = s^2 + 4 - 4$

$$X_1 = \frac{s^2}{s^4 + 14s^2 + 24} \quad \Rightarrow \quad X_1 = \frac{s^2}{(s^2 + 2)(s^2 + 12)}$$

Descomponiendo en fracciones parciales:

$$\frac{s^2}{(s^2 + 2)(s^2 + 12)} = \frac{A(2s) + B}{s^2 + 2} + \frac{C(2s) + D}{s^2 + 12}$$

$s^2 = (2As + B)(s^2 + 12) + (2Cs + D)(s^2 + 2)$

$s^2 = 2As^3 + 24As + Bs^2 + 12B + 2Cs^3 + 4Cs + Ds^2 + 2D$

$s^2 = (2A + 2C)s^3 + (B + D)s^2 + (24A + 4C)s + (12B + 2D)$

$0 = 2A + 2C$

$1 = B + D$

$0 = 24A + 4C$

$0 = 12B + 2D$

Resolviendo el sistema A = 0 , B = -1/5 , C = 0 , D = 6/5

$$X_1 = -\frac{1/5}{s^2 + 2} + \frac{6/5}{s^2 + 12}$$

Aplicando transformada inversa:

$$\mathcal{L}^{-1}\{X_1\} = -\frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s^2+2}\right\} + \frac{6}{5}\mathcal{L}^{-1}\left\{\frac{1}{s^2+12}\right\}$$

$$x_1(t) = -\frac{1}{5\sqrt{2}}Sen\left(\sqrt{2}t\right) + \frac{6}{5\sqrt{12}}Sen\left(\sqrt{12}t\right)$$

Sustituimos X_1 en cualquiera de las dos ecuaciones:

$$X_2 = \frac{(s^2+10)X_1 - 1}{4}$$

$$X_2 = \frac{(s^2+10)}{4}\left[\frac{6}{5(s^2+12)} - \frac{1}{5(s^2+2)}\right] - \frac{1}{4}$$

$$X_2 = \frac{(s^2+10)}{20}\left[\frac{6s^2+12-s^2-12}{(s^2+12)(s^2+2)}\right] - \frac{1}{4}$$

$$X_2 = \frac{s^2(s^2+10)}{4(s^2+12)(s^2+2)} - \frac{1}{4}$$

$$X_2 = \frac{s^4+10s^2-s^4-14s^2-24}{4(s^2+12)(s^2+2)}$$

$$X_2 = \frac{-4s^2-24}{4(s^2+12)(s^2+2)}$$

$$X_2 = -\frac{s^2+6}{(s^2+12)(s^2+2)}$$

Descomponiendo en fracciones parciales:

$$\frac{s^2+6}{(s^2+2)(s^2+12)} = \frac{A(2s)+B}{s^2+2} + \frac{C(2s)+D}{s^2+12}$$

$$s^2+6 = (2As+B)(s^2+12) + (2Cs+D)(s^2+2)$$

$$s^2+6 = 2As^3+24As+Bs^2+12B+2Cs^3+4Cs+Ds^2+2D$$

$$s^2+6 = (2A+2C)s^3+(B+D)s^2+(24A+4C)s+(12B+2D)$$

$$0 = 2A+2C$$

$$1 = B+D$$

$$0 = 24A+4C$$

$$6 = 12B+2D$$

Resolviendo el sistema A = 0 , B = 2/5 , C = 0 , D = 3/5

$$X_2 = -\frac{s^2+6}{(s^2+12)(s^2+2)}$$

Aplicando transformada inversa:

$$\mathcal{L}^{-1}\{X_2\} = -\frac{2}{5}\mathcal{L}^{-1}\left\{\frac{1}{s^2+2}\right\} - \frac{3}{5}\mathcal{L}^{-1}\left\{\frac{1}{s^2+12}\right\}$$

$$x_2(t) = -\frac{2}{5\sqrt{2}}Sen\left(\sqrt{2}t\right) - \frac{3}{5\sqrt{12}}Sen\left(\sqrt{12}t\right)$$

Finalmente las ecuaciones de los desplazamientos verticales para las masas son:

$$\boxed{\boldsymbol{x_1(t) = -\frac{\sqrt{2}}{10}Sen\left(\sqrt{2}t\right) + \frac{\sqrt{3}}{5}Sen\left(2\sqrt{3}t\right)}}$$

$$\boxed{\boldsymbol{x_2(t) = -\frac{\sqrt{2}}{5}Sen\left(\sqrt{2}t\right) - \frac{\sqrt{3}}{10}Sen\left(2\sqrt{3}t\right)}}$$

CIRCUITOS ELÉCTRICOS

1) Determine la corriente i(t) en un circuito RLC serie, cuando L = 1 h , R = 0 Ω , C = 10^{-4}F y el voltaje aplicado es:

$$\boldsymbol{E(t)}\begin{cases} \mathbf{100\ Sen(10t)} \ ; & \mathbf{0 \leq t} < \pi \\ \mathbf{0} \ ; & \boldsymbol{t \geq \pi} \end{cases}$$

La ecuación para este circuito es:

$$L\frac{di}{dt} + iR + \frac{1}{C}\int i\,dt = E(t)$$

Entonces tenemos:

$$\frac{di}{dt} + 10^4\int i\,dt = (\mathcal{u}_0 - \mathcal{u}_\pi)100\,Sen(10t)$$

$$i' + 10^4\int i\,dt = 100\mathcal{u}_0 Sen(10t) - 100\mathcal{u}_\pi Sen(10t)$$

Hay que desfasar la función Sen(10t)

$$Sen[10(t+\pi-\pi)] = Sen[10(t+\pi) - 10\pi]$$

$$Sen[10(t+\pi-\pi)] = Sen[10(t+\pi)]Cos(10\pi) + Cos[10(t+\pi)]Sen(10\pi)$$

$$Sen[10(t+\pi-\pi)] = Sen[10(t+\pi)]$$

Entonces:

$$i' + 10^4\int i\,dt = 100\mathcal{u}_0 Sen(10t) - 100\mathcal{u}_\pi Sen[10(t+\pi)]$$

Aplicando transformada de Laplace:

$$sI - i(0) + 10^4\frac{I}{s} = \frac{1000}{s^2+100} - \frac{1000e^{-\pi s}}{s^2+100}$$

Sabemos que i(0) = 0

$$s^2I + 10^4I = \frac{1000s}{s^2+100} - \frac{1000s}{s^2+100}e^{-\pi s}$$

$$I = \frac{1000s}{(s^2+100)(s^2+1000)} - \frac{1000s}{(s^2+100)(s^2+1000)}e^{-\pi s}$$

Aplicando transformada inversa:

$$\mathcal{L}^{-1}\{I\} = 10\mathcal{L}^{-1}\left\{\frac{100s}{(s^2+100)(s^2+1000)}\right\} - 10\mathcal{L}^{-1}\left\{\frac{100s}{(s^2+100)(s^2+1000)}e^{-\pi s}\right\}$$

Aplicando convolución:

$$\mathcal{L}^{-1}\left\{\frac{100s}{(s^2+100)(s^2+1000)}\right\}=\mathcal{L}^{-1}\left\{\frac{s}{s^2+100}*\frac{100}{s^2+1000}\right\}$$

$$\int_0^t Cos(10x)Sen[100(t-x)]dx$$

$$\frac{1}{2}\int_0^t [Sen(10x+100t-100x)-Sen(10x-100t+100x)]dx$$

$$\frac{1}{2}\int_0^t [Sen(-90x+100t)-Sen(110x-100t)]dx$$

$$\frac{1}{2}\left[\frac{1}{90}Cos(-90x+100t)+\frac{1}{110}Cos(110x-100t)\right]_0^t$$

$$\frac{1}{2}\left[\frac{1}{90}Cos(10t)+\frac{1}{110}Cos(10t)-\frac{1}{90}Cos(100t)-\frac{1}{110}Cos(100t)\right]$$

Entonces:

$$i(t)=10\left(\frac{1}{2}\right)\left[\frac{1}{90}Cos(10t)+\frac{1}{110}Cos(10t)-\frac{1}{90}Cos(100t)-\frac{1}{110}Cos(100t)\right]-$$

$$10\left(\frac{1}{2}\right)u(t-\pi)\left[\frac{1}{90}Cos[10(t-\pi)]+\frac{1}{110}Cos[10(t-\pi)]-\frac{1}{90}Cos[100(t-\pi)]-\frac{1}{110}Cos[100(t-\pi)]\right]$$

Luego:

$$Cos(10t-10\pi)=Cos(10t)Cos(10\pi)+Sen(10t)Sen(10\pi)$$

$$Cos(10t-10\pi)=Cos(10t)$$

Así mismo:

$$Cos(100t-100\pi)=Cos(100t)$$

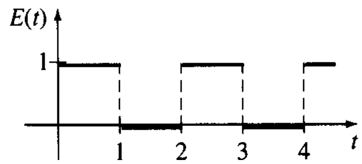
Finalmente:

$$i(t)=\frac{1}{18}Cos(10t)+\frac{1}{22}Cos(10t)-\frac{1}{18}Cos(100t)-\frac{1}{22}Cos(100t)$$

$$-u(t-\pi)\left[\frac{1}{18}Cos(10t)+\frac{1}{22}Cos(10t)-\frac{1}{18}Cos(100t)-\frac{1}{22}Cos(100t)\right]$$

$$\boxed{\boldsymbol{i(t)=\frac{1}{18}[Cos(10t)-Cos(100t)]+\frac{1}{22}[Cos(10t)-Cos(100t)]-u(t-\pi)\left[\frac{1}{18}[Cos(10t)-Cos(100t)]\right]}}$$

2) En un circuito LR, determine la corriente i(t) para cualquier tiempo t, sabemos, que cuando i(0) = 0 y E(t) es la función onda cuadrada que muestra la figura. Luego suponga que L = 1 y R = 1 y determine i(t) para el intervalo 0 ≤ t < 4.



Planteando la ecuación:

$$L\frac{di}{dt} + Ri = E(t)$$

Aplicando la transformada de Laplace:

$$L[sI - i(0)] + RI = \frac{1}{1-e^{-2s}}\left(\int_0^1 (1)e^{-st}\,dt + \int_1^2 (0)e^{-st}\,dt\right)$$

$$LsI + RI = \frac{1}{1-e^{-2s}}\left(-\frac{1}{s}e^{-st}\Big|_0^1\right) \quad \Rightarrow \quad LsI + RI = \frac{1}{1-e^{-2s}}\left(-\frac{1}{s}e^{-s} + \frac{1}{s}\right)$$

$$LsI + RI = \frac{1}{1-e^{-2s}}\left(\frac{1-e^{-s}}{s}\right) \quad \Rightarrow \quad LsI + RI = \frac{1}{(1-e^{-s})(1+e^{-s})}\left(\frac{1-e^{-s}}{s}\right)$$

$$LsI + RI = \frac{1}{s(1+e^{-s})}$$

$$I(Ls + R) = \frac{1}{s(1+e^{-s})}$$

$$I = \frac{1}{s(1+e^{-s})(Ls+R)}$$

$$I = \frac{{}^{1}/_{L}}{s(1+e^{-s})\left(s + {}^{R}/_{L}\right)}$$

Para determinar la transformada de Laplace de esta función, primero emplearemos una serie geométrica.

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ \dots\dots\dots\dots$$

Si $x = e^{-s}$ *, cuando s > 0 tenemos que:*

$$\frac{1}{1+e^{-s}} = 1 - e^{-s} + e^{-2s} - e^{-3s} + \ \dots\dots\dots\dots$$

Si escribimos:

$$I = \left(\frac{1}{R}\right)\frac{\left(\frac{R}{L}+s\right)-s}{s\left(s+{}^{R}/_{L}\right)}\left[\frac{1}{(1+e^{-s})}\right]$$

$$I = \frac{1}{R(1+e^{-s})}\left[\frac{\left(\frac{R}{L}+s\right)}{s\left(s+{}^{R}/_{L}\right)}-\frac{s}{s\left(s+{}^{R}/_{L}\right)}\right]$$

$$I = \frac{1}{R}\left[\frac{1}{s}-\frac{1}{\left(s+{}^{R}/_{L}\right)}\right](1+e^{-s})$$

Entonces:

$$I = \frac{1}{R}\left[\frac{1}{s}-\frac{1}{\left(s+{}^{R}/_{L}\right)}\right](1-e^{-s}+e^{-2s}-e^{-3s}+\ \dots\dots\dots)$$

$$I = \frac{1}{R}\left(\frac{1}{s}-\frac{e^{-s}}{s}+\frac{e^{-2s}}{s}-\frac{e^{-3s}}{s}+\ \dots\dots\right)-\frac{1}{R}\left(\frac{1}{s+{}^{R}/_{L}}-\frac{e^{-s}}{s+{}^{R}/_{L}}+\frac{e^{-2s}}{s+{}^{R}/_{L}}-\frac{e^{-3s}}{s+{}^{R}/_{L}}+\ \dots\dots\right)$$

Al aplicar la forma inversa del segundo teorema de traslación a cada término de ambas series tenemos:

$$i(t) = \frac{1}{R}[1-\mathcal{u}(t-1)+\mathcal{u}(t-2)-\mathcal{u}(t-3)+\ \dots\dots]$$
$$-\frac{1}{R}\left[e^{-\frac{R}{L}t}-e^{-\frac{R}{L}(t-1)}\mathcal{u}(t-1)+e^{-\frac{R}{L}(t-2)}\mathcal{u}(t-2)-e^{-\frac{R}{L}(t-3)}\mathcal{u}(t-3)+\ \dots\dots\right]$$

O, lo que es lo mismo:

$$\boxed{\boldsymbol{i(t) = \frac{1}{R}\left(1-e^{-\frac{R}{L}t}\right)+\frac{1}{R}\sum_{n=1}^{+\infty}(-1)^n\left(1-e^{-\frac{R}{L}(t-n)}\right)u(t-n)}}$$

Finalmente, encontrando i(t) para 0 ≤ t < 4 tenemos:

$$\boxed{\boldsymbol{i(t) = 1-e^{-t}-\left[1-e^{-(t-1)}\right]u(t-1)+\left[1-e^{-(t-2)}\right]u(t-2)-\left[1-e^{-(t-3)}\right]u(t-3)}}$$

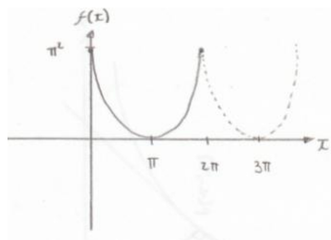
O, lo que es lo mismo:

$$\boxed{\boldsymbol{i(t) = \begin{cases} 1-e^{-t} & 0\le t<1 \\ -e^{t}+e^{-(t-1)} & 1\le t<2 \\ 1-e^{-t}+e^{-(t-1)}-e^{-(t-2)} & 2\le t<3 \\ e^{-t}+e^{-(t-1)}-e^{-(t-2)}+e^{-(t-3)} & 3\le t<4 \end{cases}}}$$

SERIES DE FOURIER

1) Obtenga la expansión en serie de Fourier de la función periódica $f(t)$ de periodo 2π definida sobre el periodo $0 \le t \le 2\pi$ por $f(t) = (\pi - t)^2$ y de allí demostrar que $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n Cos\left(\frac{n\pi}{l}t\right) + b_n Sen\left(\frac{n\pi}{l}t\right)\right] \quad ; \quad T = 2l = 2\pi$$



Dado que $f(t)$ es par:

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(t)dt \quad ; \quad a_n = \frac{1}{l}\int_{-l}^{l} f(t)Cos\left(\frac{n\pi}{l}t\right)dt \quad ; \quad b_n = 0$$

Encontrando a_0

$$a_0 = \frac{2}{\pi}\int_{0}^{\pi} (\pi - t)^2 dt = \frac{2}{\pi}\left[-\frac{(\pi - t)^3}{3}\right]_0^{\pi} = -\frac{2}{\pi}\left(-\frac{\pi^3}{3}\right) = \frac{2}{3}\pi^2$$

Encontrando a_n

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} (\pi - t)^2 Cos(nt)dt$$

$u = (\pi - t)^2 \qquad du = -2(\pi - t)dt$

$dv = Cos(nt)dt \qquad v = \frac{1}{n}Sen(nt)$

$$a_n = \frac{2}{\pi}\left[\frac{(\pi - t)^2}{n}Sen(nt) - \int -\frac{2}{n}(\pi - t)Sen(nt)dt\right] = \frac{2}{\pi}\left[\frac{(\pi - t)^2}{n}Sen(nt) + \frac{2}{n}\int [\pi Sen(nt) - t\,Sen(nt)]\,dt\right]$$

$$a_n = \frac{2}{\pi}\left[\frac{(\pi - t)^2}{n}Sen(nt) - \frac{2\pi}{n^2}Cos(nt) - \frac{2}{n}\int t\,Sen(nt)\,dt\right]$$

$u = t \qquad du = dt$

$dv = Sen(nt)dt \qquad v = -\frac{1}{n}Cos(nt)$

$$a_n = \frac{2}{\pi}\left[\frac{(\pi - t)^2}{n}Sen(nt) - \frac{2\pi}{n^2}Cos(nt) - \frac{2}{n}\left[-\frac{t}{n}Cos(nt) + \frac{1}{n}\int Cos(nt)dt\right]\right]$$

$$a_n = \frac{2}{\pi}\left[\frac{(\pi - t)^2}{n}Sen(nt) - \frac{2\pi}{n^2}Cos(nt) + \frac{2t}{n^2}Cos(nt) - \frac{2}{n^3}Sen(nt)\right]_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left[-\frac{2\pi}{n^2}Cos(n\pi) + \frac{2\pi}{n^2}Cos(n\pi) + \frac{2\pi}{n^2}\right] = \frac{4}{n^2}$$

Reemplazando:

$$\boxed{\boldsymbol{f(t) = \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty}\frac{4}{n^2}Cos\,(nt)}}$$

Si $t = \pi$, entonces:

$$f(\pi) = \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty}\frac{4}{n^2}Cos\,(n\pi) = \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n$$

$$0 = \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n$$

$$-\frac{1}{3}\pi^2 = 4\sum_{n=1}^{\infty}\frac{1}{n^2}(-1)^n$$

Multiplicando por -1, finalmente tenemos:

$$\boxed{\boldsymbol{\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}}}$$

2) Con respecto a la función $f(t)=\begin{cases}\pi^2, & -\pi<t<0\\ (t-\pi)^2, & 0<t<\pi\end{cases}, \quad f(t+2\pi)=f(t)$**:**

a) Pruebe que la serie de Fourier que representa la función periódica $f(t)$ es:

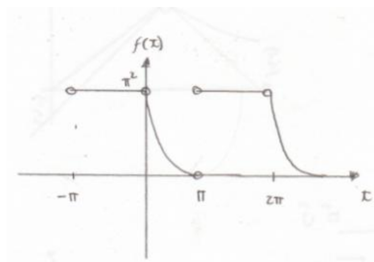
$$f(t)=\frac{2\pi^2}{3}+\sum_{n=1}^{\infty}\left[\frac{2}{n^2}\cos(nt)+\frac{(-1)^n}{n}\pi sen(nt)\right]-\frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{sen((2n-1)t)}{(2n-1)^3}\right]$$

b) Utilice este resultado para probar que:

i) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{12}$

Para a)

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n Cos\left(\frac{n\pi}{l}t\right)+b_n Sen\left(\frac{n\pi}{l}t\right)\right] \quad ; \quad T=2l=2\pi$$



Dado que $f(t)$ no es impar y tampoco par:

$$a_0=\frac{1}{l}\int_{-l}^{l}f(t)dt \quad ; \quad a_n=\frac{1}{l}\int_{-l}^{l}f(t)Cos\left(\frac{n\pi}{l}t\right)dt \quad ; \quad b_n=\frac{1}{l}\int_{-l}^{l}f(t)Sen\left(\frac{n\pi}{l}t\right)dt$$

Encontrando a_0

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)dt=\frac{1}{\pi}\left[\int_{-\pi}^{0}\pi^2dt+\int_{0}^{\pi}(t-\pi)^2dt\right]=\frac{1}{\pi}\left[[\pi^2t]_{-\pi}^{0}+\left[\frac{(t-\pi)^3}{3}\right]_0^{\pi}\right]$$

$$a_0=\frac{1}{\pi}\left[\pi^3+\frac{\pi^3}{3}\right]=\frac{4\pi^2}{3}$$

Encontrando a_n

$$a_n = \frac{1}{\pi}\int_{-l}^{l} f(t)Cos\left(\frac{n\pi}{l}t\right)dt = \frac{1}{\pi}\left[\int_{-\pi}^{0} \pi^2 Cos(nt)dt + \int_{0}^{\pi} (t-\pi)^2 Cos(nt)dt\right]$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} \pi^2 Cos(nt)dt + \int_{0}^{\pi} (t-\pi)^2 Cos(nt)dt\right] = \frac{1}{\pi}\left[\frac{\pi^2}{n}[Sen(nt)]_{-\pi}^{0} \underset{-\pi}{0} + \underbrace{\int_{0}^{\pi} (t-\pi)^2 Cos(nt)dt}_{\substack{Resuelta\ \ en\ el \\ ejercicio\ \ anterior}}\right]$$

$$a_n = \frac{1}{\pi}\left[\frac{\pi^2}{n}Sen(n\pi) + \frac{2\pi}{n^2}\right] = \frac{2}{n^2}$$

Encontrando b_n

$$b_n = \frac{1}{\pi}\int_{-l}^{l} f(t)Sen\left(\frac{n\pi}{l}t\right)dt = \frac{1}{\pi}\left[\int_{-\pi}^{0} \pi^2 Sen(nt)dt + \int_{0}^{\pi} (t-\pi)^2 Sen(nt)dt\right]$$

$$b_n = \frac{1}{\pi}\left[-\frac{\pi^2}{n}[Cos(nt)]_{-\pi}^{0} + \int_{0}^{\pi} (t-\pi)^2 Sen(nt)dt\right]$$

$$\int (t-\pi)^2 Sen(nt)dt$$

$$u = (t-\pi)^2 \qquad du = 2(t-\pi)dt$$

$$dv = Sen(nt)dt \qquad v = -\frac{1}{n}Cos(nt)$$

$$= -\frac{(t-\pi)^2}{n}Cos(nt) + \frac{2}{n}\int (t-\pi)Cos(nt)dt = -\frac{(t-\pi)^2}{n}Cos(nt) + \frac{2}{n}\left[\int t\, Cos(nt)dt - \pi\int Cos(nt)dt\right]$$

$$= -\frac{(t-\pi)^2}{n}Cos(nt) + \frac{2}{n}\int t\, Cos(nt)dt - \frac{2\pi}{n^2}Sen(nt)$$

$$u = t \qquad du = dt$$

$$dv = Cos(nt)dt \qquad v = \frac{1}{n}Sen(nt)$$

$$= -\frac{(t-\pi)^2}{n}Cos(nt) + \frac{2}{n}\left[\frac{t}{n}Sen(nt) - \frac{1}{n}\int Sen(nt)dt\right] - \frac{2\pi}{n^2}Sen(nt)$$

$$= -\frac{(t-\pi)^2}{n}Cos(nt) + \frac{2t}{n^2}Sen(nt) + \frac{2}{n^3}Cos(nt) - \frac{2\pi}{n^2}Sen(nt)$$

$$b_n = \frac{1}{\pi}\left[-\frac{\pi^2}{n}[Cos(nt)]_{-\pi}^{0} + \left[-\frac{(t-\pi)^2}{n}Cos(nt) + \frac{2t}{n^2}Sen(nt) + \frac{2}{n^3}Cos(nt) - \frac{2\pi}{n^2}Sen(nt)\right]_{0}^{\pi}\right]$$

$$b_n = \frac{1}{\pi}\left[\left[-\frac{\pi^2}{n} + \frac{\pi^2}{n}Cos(n\pi)\right] + \left[\frac{2}{n^3}Cos(n\pi) + \frac{\pi^2}{n} - \frac{2}{n^3}\right]\right]$$

$$b_n = \frac{1}{\pi}\left[-\frac{\pi^2}{n} + \frac{\pi^2}{n}Cos(n\pi) + \frac{2}{n^3}Cos(n\pi) + \frac{\pi^2}{n} + \frac{2}{n^3}\right] = \frac{1}{\pi}\left[\frac{\pi^2}{n}Cos(n\pi) + \frac{2}{n^3}Cos(n\pi) - \frac{2}{n^3}\right]$$

$$b_n = \frac{1}{\pi}\left[\frac{\pi^2}{n}(-1)^n + \frac{2}{n^3}(-1)^n - \frac{2}{n^3}\right]$$

Reemplazando:

$$f(t) = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{2}{n^2}Cos\,(nt) + \left[\frac{\pi}{n}(-1)^n + \frac{2}{\pi n^3}(-1)^n - \frac{2}{\pi n^3}\right]Sen\,(nt)\right]$$

$$f(t) = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{2}{n^2}Cos\,(nt) + \frac{(-1)^n}{n}\pi\,Sen\,(nt)\right] + \sum_{n=1}^{\infty}\left[\frac{2(-1)^n}{\pi n^3} - \frac{2}{\pi n^3}\right]Sen\,(nt)$$

$Hacemos\ un\ cambio\ de\ variable\quad \boldsymbol{n = 2m - 1},\qquad entonces:$

$Cuando\ \ n = 1\ \ \Rightarrow\ \ m = 2(1) - 1 = 1\ \ y\ cuando\ n = \infty\ \ \Rightarrow\ \ m = 2(\infty) - 1 = \infty$

$$f(t) = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{2}{n^2}Cos\,(nt) + \frac{(-1)^n}{n}\pi\,Sen\,(nt)\right] + \sum_{m=1}^{\infty}\left[\frac{2(-1)^{\overbrace{2m-1}^{Siempre\ \ es\ impar}}}{\pi(2m-1)^3} - \frac{2}{\pi(2m-1)^3}\right]Sen\,[(2m-1)t]$$

$$f(t) = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{2}{n^2}Cos\,(nt) + \frac{(-1)^n}{n}\pi\,Sen\,(nt)\right] + \sum_{m=1}^{\infty}\left[-\frac{2}{\pi(2m-1)^3} - \frac{2}{\pi(2m-1)^3}\right]Sen\,[(2m-1)t]$$

$$f(t) = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{2}{n^2}Cos\,(nt) + \frac{(-1)^n}{n}\pi\,Sen\,(nt)\right] + \sum_{m=1}^{\infty}\left[-\frac{4}{\pi(2m-1)^3}\right]Sen\,[(2m-1)t]$$

Finalmente tenemos:

$$\boxed{\boldsymbol{f(t) = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{2}{n^2}Cos\,(nt) + \frac{(-1)^n}{n}\pi\,Sen\,(nt)\right] - \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{Sen\,\left((2n-1)t\right)}{(2n-1)^3}\right]}}$$

<u>Para b)</u>

Si t = π

$$f(\pi) = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{2}{n^2}Cos\,(n\pi) + \frac{(-1)^n}{n}\pi\,Sen\,(n\pi)\right] - \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{Sen\,\left((2n-1)\pi\right)}{(2n-1)^3}\right]$$

Generando términos:

$$f(\pi) = \frac{2\pi^2}{3} + \left[[2Cos\ (\pi) - \pi\ Sen\ (\pi)] + \left[\frac{2}{2^2} Cos\ (2\pi) + \frac{1}{2}\pi\ Sen\ (2\pi) \right] + \left[\frac{2}{3^2} Cos\ (3\pi) + \frac{1}{3}\pi\ Sen\ (3\pi) \right] + \ \dots\dots \right]$$

$$-\frac{4}{\pi}\left[[Sen\ (\pi)] + \left[\frac{Sen\ (3\pi)}{(3)^3} \right] + \left[\frac{Sen\ (5\pi)}{(5)^3} \right] + \ \dots\dots \right]$$

$$f(\pi) = \frac{2\pi^2}{3} + 2\left(-1 + \frac{1}{2^2} - \frac{1}{3^2} + \ \dots\dots \right)\ ;\ como\ f(\pi) es\ un\ punto\ de\ descontinuidad, entonces\ f(\pi) = \frac{1}{2}[f(t^+) + f(t^-)]$$

$$\frac{\pi^2}{2} = \frac{2\pi^2}{3} + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \quad \Rightarrow \quad -\frac{\pi^2}{6} = 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$Multiplicando\ por - 1\ tenemos:$

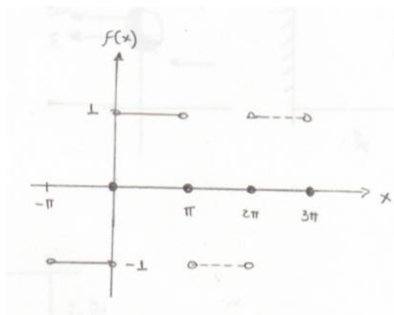
$$\boxed{\boldsymbol{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}}}$$

3) Determine la representación en serie de Fourier de la función periódica f(t) de periodo 2π definida por la siguiente regla de correspondencia: $\boldsymbol{f(t) = \begin{cases} -1 & ; -\pi < x < 0 \\ 1 & ; \quad 0 < x < \pi \\ 0 & ; x = 0 ; x = \pi \end{cases}}$

Determine a que converge la serie:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)}$$

Haciendo un bosquejo de la gráfica:



Podemos que f(t) es impar, entonces:

$$b_n = \frac{1}{l}\int_{-l}^{l} f(t)Sen\left(\frac{n\pi}{l}t\right)dt$$

Encontrando b_n

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} Sen(nt)f(t)dt = \frac{2}{\pi}\int_{0}^{\pi} Sen(nt)f(t)dt$$

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} Sen(nt)dt$$

$$b_n = \frac{2}{\pi}\left[-\frac{1}{n}Cos(nt)\right]_0^{\pi}$$

$$b_n = \frac{2}{\pi}\left[\frac{1}{n} - \frac{1}{n}Cos(n\pi)\right]$$

Reemplazando:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n Cos(nt) + b_n Sen(nt)]$$

$$f(t) = \sum_{n=1}^{\infty} \frac{2}{\pi}\left[\frac{1}{n} - \frac{1}{n}Cos(n\pi)\right]Sen(nt)$$

Generando términos:

$$f(t)=\frac{2}{\pi}\left[[1-Cos(\pi)]Sen(t)+\left[\frac{1}{2}-\frac{1}{2}Cos(2\pi)\right]Sen(2t)+\left[\frac{1}{3}-\frac{1}{3}Cos(3\pi)\right]Sen(3t)+\ \dots\dots\dots.\right]$$

$$f(t)=\frac{2}{\pi}\left[[1+1]Sen(t)+\left[\frac{1}{2}-\frac{1}{2}\right]Sen(2t)+\left[\frac{1}{3}+\frac{1}{3}\right]Sen(3t)+\ \dots\dots\dots.\right]$$

$$f(t)=\frac{2}{\pi}\left[2\ Sen(t)+\frac{2}{3}Sen(3t)+\frac{2}{5}Sen(5t)+\ \dots\dots\dots\right]$$

$$f(t)=\frac{4}{\pi}\left[\ Sen(t)+\frac{1}{3}Sen(3t)+\frac{1}{5}Sen(5t)+\ \dots\dots\dots\right]$$

Expresando en notación de sumatoria, por lo tanto la representación en serie de Fourier es:

$$\boxed{\boldsymbol{f(t)=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}Sen[(2n-1)t]}}$$

Encontrando la convergencia:

Si t = π/2

$$f(\pi/2)=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}Sen\left[(2n-1)\frac{\pi}{2}\right]$$

Generando términos:

$$f(\pi/2)=\frac{4}{\pi}\left[Sen\left(\frac{\pi}{2}\right)+\frac{1}{3}Sen\left(\frac{3\pi}{2}\right)+\frac{1}{5}Sen\left(\frac{5\pi}{2}\right)+\ \dots\dots\dots\right]$$

$$f(\pi/2)=\frac{4}{\pi}\left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ \dots\dots\dots\right]$$

Si notamos en el gráfico de f(π/2) = 1 , entonces:

$$1=\frac{4}{\pi}\left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ \dots\dots\dots\right]$$

Expresando en notación de sumatoria:

$$1=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)}$$

Entonces:

$$\boxed{\boldsymbol{\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)}=\frac{\pi}{4}}}$$

EXTENSIONES PARES E IMPARES PERIÓDICAS DE UNA SERIE DE FOURIER

1) Con respecto a la función f, definida por: $\boldsymbol{f(x) = \begin{cases} 0.1x & ;\ 0 \leq x < 10 \\ 0.1(10 - x); & 10 \leq x < 20 \end{cases}}$

determine:

a) La correspondiente serie impar de medio rango.

b) La suma de la serie numérica:

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

Para a)

$a_0 = a_n = 0$

Entonces:

$$b_n = \frac{1}{20}\int_{-20}^{20} f(t) Sen\left(\frac{n\pi}{20}t\right) dt = \frac{1}{10}\int_{0}^{20} f(t) Sen\left(\frac{n\pi}{20}t\right) dt$$

$$b_n = \frac{1}{10}\left[\int_{0}^{10} 0.1t\, Sen\left(\frac{n\pi}{20}t\right) dt + \int_{10}^{20} 0.1(10-t)\, Sen\left(\frac{n\pi}{20}t\right) dt\right]$$

Integrando en forma general, tenemos:

$$\int (at+b)\, Sen\left(\frac{n\pi}{20}t\right) dt$$

Por partes:

$$u = at + b \quad \Rightarrow \quad du = a\, dt$$

$$dv = Sen\left(\frac{n\pi}{20}t\right) dt \quad \Rightarrow \quad v = -\frac{20}{n\pi} Cos\left(\frac{n\pi}{20}t\right)$$

$$\int (at+b)\, Sen\left(\frac{n\pi}{20}t\right) dt = -\frac{20(at+b)}{n\pi} Cos\left(\frac{n\pi}{20}t\right) + \frac{20a}{n\pi}\int Cos\left(\frac{n\pi}{20}t\right) dt$$

$$\int (at+b)\, Sen\left(\frac{n\pi}{20}t\right) dt = -\frac{20}{n\pi}(at+b) Cos\left(\frac{n\pi}{20}t\right) + \left(\frac{20}{n\pi}\right)^2 a\, Sen\left(\frac{n\pi}{20}t\right)$$

Reemplazando:

$$\int_0^{10} 0.1t\,Sen\left(\frac{n\pi}{20}t\right)dt = \left[-\frac{20}{n\pi}(0.1t)Cos\left(\frac{n\pi}{20}t\right)+\left(\frac{20}{n\pi}\right)^2(0.1)Sen\left(\frac{n\pi}{20}t\right)\right]_0^{10}$$

$$\boldsymbol{\int_0^{10} 0.1t\,Sen\left(\frac{n\pi}{20}t\right)dt = \frac{1}{10}\left[-\frac{200}{n\pi}Cos\left(\frac{n\pi}{2}\right)+\left(\frac{20}{n\pi}\right)^2 Sen\left(\frac{n\pi}{2}\right)\right]}$$

$$\int_{10}^{20}(1-0.1t)\,Sen\left(\frac{n\pi}{20}t\right)dt = \left[-\frac{20}{n\pi}(1-0.1t)Cos\left(\frac{n\pi}{20}t\right)-\left(\frac{20}{n\pi}\right)^2(0.1)Sen\left(\frac{n\pi}{20}t\right)\right]_{10}^{20}$$

$$\int_{10}^{20}(1-0.1t)\,Sen\left(\frac{n\pi}{20}t\right)dt = \frac{1}{10}\left[-\frac{20}{n\pi}(10-t)Cos\left(\frac{n\pi}{20}t\right)-\left(\frac{20}{n\pi}\right)^2 Sen\left(\frac{n\pi}{20}t\right)\right]_{10}^{20}$$

$$\boldsymbol{\int_{10}^{20}(1-0.1t)\,Sen\left(\frac{n\pi}{20}t\right)dt = \frac{1}{10}\left[\frac{200}{n\pi}Cos(n\pi)-\left(\frac{20}{n\pi}\right)^2 Sen(n\pi)+\left(\frac{20}{n\pi}\right)^2 Sen\left(\frac{n\pi}{2}\right)\right]}$$

Entonces b_n:

$$b_n = \frac{1}{100}\left[-\frac{200}{n\pi}Cos\left(\frac{n\pi}{2}\right)+\left(\frac{20}{n\pi}\right)^2 Sen\left(\frac{n\pi}{2}\right)+\frac{200}{n\pi}Cos(n\pi)-\left(\frac{20}{n\pi}\right)^2 Sen(n\pi)+\left(\frac{20}{n\pi}\right)^2 Sen\left(\frac{n\pi}{2}\right)\right]$$

Podemos notar que $Sen(n\pi)=0$, entonces:

$$b_n = \frac{1}{100}\left[-\frac{200}{n\pi}Cos\left(\frac{n\pi}{2}\right)+\left(\frac{20}{n\pi}\right)^2 Sen\left(\frac{n\pi}{2}\right)+\frac{200}{n\pi}Cos(n\pi)+\left(\frac{20}{n\pi}\right)^2 Sen\left(\frac{n\pi}{2}\right)\right]$$

Luego:

$$f(t)=\sum_{n=1}^{\infty} b_n Sen\left(\frac{n\pi}{l}t\right)$$

$$f(t)=\sum_{n=1}^{\infty}\frac{1}{100}\left[-\frac{200}{n\pi}Cos\left(\frac{n\pi}{2}\right)+\left(\frac{20}{n\pi}\right)^2 Sen\left(\frac{n\pi}{2}\right)+\frac{200}{n\pi}Cos(n\pi)+\left(\frac{20}{n\pi}\right)^2 Sen\left(\frac{n\pi}{2}\right)\right]Sen\left(\frac{n\pi}{20}t\right)$$

Si n es par, entonces $Sen\left(\frac{n\pi}{2}\right)=0$ y $Cos(n\pi)=1$ además cuando n es impar $Cos\left(\frac{n\pi}{2}\right)=0$ y $Cos(n\pi)=-1$

Por lo tanto:

$$b_n = \begin{cases} \frac{1}{100}\left[-\frac{200}{n\pi}Cos\left(\frac{n\pi}{2}\right)+\frac{200}{n\pi}\right] & ;\ n\ es\ par \\ \frac{1}{100}\left[-\frac{200}{n\pi}+2\left(\frac{20}{n\pi}\right)^2 Sen\left(\frac{n\pi}{2}\right)\right] & ;n\ es\ impar \end{cases}$$

Luego expansión impar de medio rango de f(t) es:

$$\boxed{\boldsymbol{f(t) = \frac{1}{100}\sum_{n=1}^{\infty}\left[-\frac{200}{2n\pi}Cos\left(\frac{2n\pi}{2}\right)+\frac{200}{2n\pi}\right]Sen\left(\frac{2n\pi}{20}t\right)+\frac{1}{100}\sum_{n=1}^{\infty}\left[-\frac{200}{(2n-1)\pi}+2\left[\frac{20}{(2n-1)\pi}\right]^2 Sen\left[\frac{(2n-1)\pi}{2}\right]\right]Sen\left[\frac{(2n-1)\pi}{20}t\right]}}$$

<u>Para b)</u>

Si t = 10

$$f(10) = \frac{1}{100}\sum_{n=1}^{\infty}\left[-\frac{200}{2n\pi}Cos\left(\frac{2n\pi}{2}\right)+\frac{200}{2n\pi}\right]Sen(n\pi)+\frac{1}{100}\sum_{n=1}^{\infty}\left[-\frac{200}{(2n-1)\pi}+2\left[\frac{20}{(2n-1)\pi}\right]^2 Sen\left[\frac{(2n-1)\pi}{2}\right]\right]Sen\left[\frac{(2n-1)\pi}{2}\right]$$

Sabemos que $Sen(n\pi) = 0$ *, entonces:*

$$\frac{1}{2} = \frac{1}{100}\sum_{n=1}^{\infty}\left[-\frac{200}{(2n-1)\pi}+2\left[\frac{20}{(2n-1)\pi}\right]^2 Sen\left[\frac{(2n-1)\pi}{2}\right]\right]Sen\left[\frac{(2n-1)\pi}{2}\right]$$

$$\frac{1}{2} = \frac{1}{100}\sum_{n=1}^{\infty}\left[-\frac{200}{(2n-1)\pi}+2\left[\frac{20}{(2n-1)\pi}\right]^2 (-1)^{n+1}\right](-1)^{n+1}$$

$$\frac{1}{2} = \frac{1}{100}\left[\frac{200}{\pi}\sum_{n=1}^{\infty}\left[-\frac{(-1)^{n+1}}{(2n-1)}\right]+\sum_{n=1}^{\infty}2\left[\frac{20}{(2n-1)\pi}\right]^2(-1)^{2(n+1)}\right]$$

Como $2(n+1)$ *siempre es par, entonces:*

$$\frac{1}{2} = \frac{1}{100}\left[\frac{200}{\pi}\sum_{n=1}^{\infty}\left[-\frac{(-1)^{n+1}}{(2n-1)}\right]+\sum_{n=1}^{\infty}2\frac{400}{(2n-1)^2\pi^2}\right]$$

$$50 = -\frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)}+\frac{800}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

En el ejercicio anterior demostramos que:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)} = \frac{\pi}{4}$$

Entonces:

$$50 = -\frac{200}{\pi}\left(\frac{\pi}{4}\right)+\frac{800}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

$$100 = \frac{800}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

Finalmente:

$$\boxed{\boldsymbol{\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}}}$$

2) Con respecto a la función f definida por $f(x) = x(\pi - x), x \epsilon (0, \pi)$

a) Establezca la correspondiente expansión impar de medio rango de la función f

b) Determine la suma de la serie numérica:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3}$$

Para a)

$a_0 = a_n = 0$

Entonces:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t(\pi - t)Sen(nt)dt = \frac{2}{\pi}\int_{0}^{\pi} x(\pi - x)Sen(nt)dt$$

$$b_n = \frac{2}{\pi}\left[\pi\int_{0}^{\pi} t\,Sen(nt)dt - \int_{0}^{\pi} t^2\,Sen(nt)dt\right]$$

Integrando por partes:

$u = t \quad \Rightarrow \quad du = dt \qquad ; \qquad w = t^2 \quad \Rightarrow \quad dw = 2t\,dt$

$dv = Sen(nt)dt \quad \Rightarrow \quad v = -\frac{1}{n}Cos(nt)$

Entonces:

La primera integral nos queda:

$$\int t\,Sen(nt)dt = -\frac{t}{n}Cos(nt) + \frac{1}{n}\int Cos(nt)dt$$

$$\int t\,Sen(nt)dt = -\frac{t}{n}Cos(nt) + \frac{1}{n^2}Sen(nt)$$

La segunda integral nos queda:

$$\int t^2\,Sen(nt)dt = -\frac{t^2}{n}Cos(nt) + \frac{2}{n}\int t\,Cos(nt)dt$$

$$\int t^2\,Sen(nt)dt = -\frac{t^2}{n}Cos(nt) + \frac{2}{n}\left[\frac{t}{n}Sen(nt) - \frac{1}{n}\int Sen(nt)dt\right]$$

$$\int t^2\,Sen(nt)dt = -\frac{t^2}{n}Cos(nt) + \frac{2t}{n^2}Sen(nt) + \frac{2}{n^3}Cos(nt)$$

Reemplazando:

$$b_n = \frac{2}{\pi}\left[\pi\left[-\frac{t}{n}Cos(nt) + \frac{1}{n^2}Sen(nt)\right]_0^{\pi} - \left[-\frac{t^2}{n}Cos(nt) + \frac{2t}{n^2}Sen(nt) + \frac{2}{n^3}Cos(nt)\right]_0^{\pi}\right]$$

$$b_n = \frac{2}{\pi}\left[\pi\left[-\frac{\pi}{n}Cos(n\pi)\right] - \left[-\frac{\pi^2}{n}Cos(n\pi) + \frac{2}{n^3}Cos(n\pi) - \frac{2}{n^3}Cos(0)\right]\right]$$

$$b_n = \frac{2}{\pi}\left[-\frac{\pi^2}{n}Cos(n\pi) + \frac{\pi^2}{n}Cos(n\pi) - \frac{2}{n^3}Cos(n\pi) + \frac{2}{n^3}\right]$$

$$b_n = \frac{4}{\pi}\left[\frac{1}{n^3} - \frac{1}{n^3}Cos(n\pi)\right]$$

Entonces:

$$f(t) = \sum_{n=1}^{\infty} b_n Sen\left(\frac{n\pi}{l}t\right)$$

$$f(t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{n^3}[1 - Cos(n\pi)]\right]Sen\,(nt)$$

Generando términos:

$$f(t) = \frac{4}{\pi}\left[[1 - Cos(\pi)]Sen(t) + \frac{1}{2^3}[1 - Cos(2\pi)]Sen(2t) + \frac{1}{3^3}[1 - Cos(3\pi)]Sen(3t) + \ \ldots\ldots\ldots\right]$$

$$f(t) = \frac{4}{\pi}\left[2\,Sen(t) + \frac{2}{3^3}Sen(3t) + \frac{2}{5^3}Sen(5t) + \ \ldots\ldots\ldots\ldots\right]$$

Finalmente la representación la expansión impar de medio rango de f, nos queda:

$$\boxed{\boldsymbol{f(t) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}Sen[(2n-1)t]}}$$

<u>Para b)</u>

Si t = π/2

$$f(\pi/2) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}Sen\left[(2n-1)\frac{\pi}{2}\right]$$

$$\frac{\pi^2}{4} = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(1)^{n+1}}{(2n-1)^3}$$

Finalmente:

$$\boxed{\boldsymbol{\sum_{n=1}^{\infty}\frac{(1)^{n+1}}{(2n-1)^3} = \frac{\pi^3}{32}}}$$

3) A la función f(x) = Sen(x) , 0 < x < π expresarla mediante un desarrollo de series de cosenos, y utilizando la serie obtenida y aplicando el teorema de convergencia de las series de Fourier determine la suma de la serie numérica:

$$\sum_{n=1}^{\infty} \frac{1}{(2n)^2 - 1}$$

<u>Para a)</u>

$b_n = 0$

Entonces:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} Sen(x)dx = \frac{2}{\pi}\int_{0}^{\pi} Sen(x)dx$$

$$a_0 = -\frac{2}{\pi}Cos(x)\Big|_0^{\pi} = -\frac{2}{\pi}(-1-1) \quad \Rightarrow \quad \boldsymbol{a_0 = \frac{4}{\pi}}$$

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} Sen(x)\ Cos(nx)dx$$

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} \frac{1}{2}[Sen(x+nx) + Sen(x-nx)]dx$$

$$a_n = \frac{1}{\pi}\left[\int_{0}^{\pi} Sen[x(1+n)]dx + \int_{0}^{\pi} Sen[x(1-n)]dx\right]$$

$$a_n = \frac{1}{\pi}\left[-\frac{1}{1+n}Cos[x(1+n)] - \frac{1}{1-n}Cos[x(1-n)]\right]_0^{\pi}$$

$$a_n = \frac{1}{\pi}\left[-\frac{1}{1+n}Cos[x(1+n)] - \frac{1}{1-n}Cos[-x(n-1)]\right]_0^{\pi}$$

Sabemos que:

$$Cos[-x(n-1)] = Cos[x(n-1)]$$

Entonces:

$$a_n = \frac{1}{\pi}\left[-\frac{1}{n+1}Cos[x(1+n)] - \frac{1}{1-n}Cos[x(n-1)]\right]_0^{\pi}$$

$$a_n = -\frac{1}{\pi}\left[\frac{1}{n+1}Cos[x(1+n)]\frac{1}{1-n}Cos[x(n-1)]\right]_0^{\pi}$$

$$a_n = -\frac{1}{\pi}\left[\frac{1}{n+1}Cos[x(1+n)] - \frac{1}{n-1}Cos[x(n-1)]\right]_0^{\pi}$$

Evaluando:

$$a_n = -\frac{1}{\pi}\left[\frac{1}{n+1}Cos[\pi(1+n)] - \frac{1}{n-1}Cos[\pi(n-1)] - \frac{1}{n+1} + \frac{1}{n-1}\right]$$

Si resolvemos:

$$Cos(\pi + \pi n) = Cos(\pi)Cos(\pi n) - Sen(\pi)Sen(\pi n)$$

$$Cos(\pi + \pi n) = -Cos(\pi n)$$

Entonces:

$$a_n = -\frac{1}{\pi}\left[-\frac{1}{n+1}Cos(\pi n) + \frac{1}{n-1}Cos(\pi n) + \frac{2}{n^2-1}\right]$$

$$a_n = -\frac{1}{\pi}\left[-\frac{1}{n+1}Cos(\pi n) + \frac{1}{n-1}Cos(\pi n) + \frac{2}{n^2-1}\right]$$

$$a_n = -\frac{1}{\pi}\left[Cos(\pi n)\left[\frac{1}{n-1} - \frac{1}{n+1}\right] + \frac{2}{n^2-1}\right]$$

$$a_n = -\frac{1}{\pi}\left[Cos(\pi n)\left[\frac{n+1-n+1}{n^2-1}\right] + \frac{2}{n^2-1}\right]$$

$$a_n = -\frac{1}{\pi}\left[Cos(\pi n)\left[\frac{2}{n^2-1}\right] + \frac{2}{n^2-1}\right]$$

$$\boldsymbol{a_n = -\frac{1}{\pi}\left[\frac{2}{n^2-1}\right][Cos(\pi n) + 1]}$$

Reemplazando:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\, Cos\left(\frac{n\pi}{l}x\right)$$

$$f(x) = \frac{2}{\pi} - \sum_{n=1}^{\infty} \frac{1}{\pi}\left[\frac{2}{n^2-1}\right][Cos(\pi n) + 1]\, Cos(nx)$$

Cuando n es par:

$$Cos(\pi n) + 1 = 2$$

Cuando n es impar:

$$Cos(\pi n) + 1 = 0$$

Entonces solamente generemos términos pares:

$$f(x) = \frac{2}{\pi} - \sum_{n=1}^{\infty} \frac{1}{\pi}\left[\frac{2}{(2n)^2 - 1}\right] 2\,Cos(2nx)$$

Finalmente la expansión par de f es:

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n)^2 - 1}\; Cos(2nx)$$

Encontrando la convergencia:

Si t = π

$$f(\pi) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n)^2 - 1}\; Cos(2\pi n)$$

Si generamos términos:

$$0 = \frac{2}{\pi} - \frac{4}{\pi}\left[Cos(2\pi) + \frac{1}{15}\; Cos(4\pi) + \frac{1}{35}\; Cos(6\pi) + \; \dots\dots\dots\right]$$

$$0 = \frac{2}{\pi} - \frac{4}{\pi}\left[1 + \frac{1}{15} + \frac{1}{35} + \; \dots\dots\dots\right]$$

Podemos notar que $Cos(2\pi n) = 1$

Entonces:

$$f(2\pi) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n)^2 - 1}$$

$$0 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n)^2 - 1}$$

$$-\frac{2}{\pi} = -\frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n)^2 - 1}$$

Finalmente tenemos que:

$$\boxed{\sum_{n=1}^{\infty} \frac{\mathbf{1}}{(\mathbf{2n})^2 - \mathbf{1}} = \frac{\mathbf{1}}{\mathbf{2}}}$$

ECUACIONES EN DERIVADAS PARCIALES

ECUACIÓN DE CALOR

La ecuación de calor es:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

Resolución de ecuación de calor

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\left(\frac{n\pi\alpha}{l}\right)^2 t}\, Sen\left(\frac{n\pi}{l}x\right)$$

Donde f(x) es la temperatura en cualquier punto de la varilla:

$$f(x) = \sum_{n=1}^{\infty} C_n\, Sen\left(\frac{n\pi}{l}x\right)$$

Y C_n se lo calcula de la siguiente forma:

$$C_n = \frac{2}{l}\int_0^l f(x) Sen\left(\frac{n\pi}{l}x\right) dx$$

1) Determine la solución de la siguiente ecuación con derivadas parciales

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \;\; ; \;\; 0 < x < 100 \;\; ; \;\; t > 0 \;\; ; u(0,t) = u(100,t) = 0 \;\; ; \;\; u(x,0) = x(100-x)$$

Determinando C_n

$$C_n = \frac{2}{100}\int_0^{100} x(100-x) Sen\left(\frac{n\pi}{100}x\right) dx \quad \Rightarrow \quad C_n = \frac{2}{100}\left[\int_0^{100} (100x - x^2) Sen\left(\frac{n\pi}{100}x\right) dx\right]$$

Integrando por partes:

$$u = 100x - x^2 \quad \Rightarrow \quad du = (100-2x)dx \qquad ; \qquad dv = Sen\left(\frac{n\pi}{100}x\right) dx \quad \Rightarrow \quad v = -\frac{100}{n\pi} Cos\left(\frac{n\pi}{100}x\right)$$

Entonces:

$$\int (100x - x^2) Sen\left(\frac{n\pi}{100}x\right) dx = -\frac{100}{n\pi}(100x - x^2) Cos\left(\frac{n\pi}{100}x\right) + \frac{100}{n\pi}\int (100-2x) Cos\left(\frac{n\pi}{100}x\right) dx$$

Integrando nuevamente por partes:

$$u = 100 - 2x \quad \Rightarrow \quad du = -2\, dx$$

$$dv = Cos\left(\frac{n\pi}{100}x\right) dx \quad \Rightarrow \quad v = \frac{100}{n\pi} Sen\left(\frac{n\pi}{100}x\right)$$

$$\int (100x - x^2) Sen\left(\frac{n\pi}{100}x\right) dx = -\frac{100}{n\pi}(100x - x^2) Cos\left(\frac{n\pi}{100}x\right) + \frac{100}{n\pi}\left[\frac{100}{n\pi}(100 - 2x) Sen\left(\frac{n\pi}{100}x\right) + \frac{200}{n\pi}\int Sen\left(\frac{n\pi}{100}x\right) dx\right]$$

$$\int (100x - x^2) Sen\left(\frac{n\pi}{100}x\right) dx = -\frac{100}{n\pi}(100x - x^2) Cos\left(\frac{n\pi}{100}x\right) + \frac{100}{n\pi}\left[\frac{100}{n\pi}(100 - 2x) Sen\left(\frac{n\pi}{100}x\right) - 2\left(\frac{100}{n\pi}\right)^2 Cos\left(\frac{n\pi}{100}x\right)\right]$$

$$\int (100x - x^2) Sen\left(\frac{n\pi}{100}x\right) dx = -\frac{100}{n\pi}(100x - x^2) Cos\left(\frac{n\pi}{100}x\right) + \left(\frac{100}{n\pi}\right)^2 (100 - 2x) Sen\left(\frac{n\pi}{100}x\right) - 2\left(\frac{100}{n\pi}\right)^3 Cos\left(\frac{n\pi}{100}x\right)$$

Evaluando:

$$\left[-\frac{100}{n\pi}x(100 - x) Cos\left(\frac{n\pi}{100}x\right) + \left(\frac{100}{n\pi}\right)^2 (100 - 2x) Sen\left(\frac{n\pi}{100}x\right) - 2\left(\frac{100}{n\pi}\right)^3 Cos\left(\frac{n\pi}{100}x\right)\right]_0^{100}$$

$$-2\left(\frac{100}{n\pi}\right)^3 Cos(n\pi) + 2\left(\frac{100}{n\pi}\right)^3$$

Ahora:

Cuando n es par tenemos que:

$$-2\left(\frac{100}{2n\pi}\right)^3 Cos(2n\pi) + 2\left(\frac{100}{2n\pi}\right)^3$$

$$-2\left(\frac{100}{2n\pi}\right)^3 + 2\left(\frac{100}{2n\pi}\right)^3 = 0$$

Cuando n es impar:

$$-2\left[\frac{100}{(2n-1)\pi}\right]^3 Cos[(2n-1)\pi] + 2\left[\frac{100}{(2n-1)\pi}\right]^3$$

$$2\left[\frac{100}{(2n-1)\pi}\right]^3 + 2\left[\frac{100}{(2n-1)\pi}\right]^3 = 4\left[\frac{100}{(2n-1)\pi}\right]^3$$

Ahora:

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\left(\frac{n\pi}{100}\right)^2 t} Sen\left(\frac{n\pi}{100}x\right)$$

Sabemos que los términos pares es igual a cero, por lo tanto solo generemos los términos impares:

$$u(x,t) = \frac{4}{50}\sum_{n=1}^{\infty} C_{(2n-1)} e^{-\left[\frac{(2n-1)\pi}{100}\right]^2 t} Sen\left[\frac{(2n-1)\pi}{100}x\right]$$

Finalmente la ecuación del calor es:

$$\boxed{\boldsymbol{u(x,t) = \frac{4}{50}\sum_{n=1}^{\infty}\left[\frac{100}{(2n-1)\pi}\right]^3 e^{-\left[\frac{(2n-1)\pi}{100}\right]^2 t} Sen\left[\frac{(2n-1)\pi}{100}x\right]}}$$

2) Determine la solución de la siguiente ecuación con derivadas parciales

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \;\; ; \;\; 0 < x < \pi \; ; \; t > 0$$

Sujeta a las siguientes condiciones:

$$u(0,t) = u(\pi,t) = 0$$

$$u(x,0) = 4\,Sen(4x)Cos(2x)$$

Determinando C_n

$$C_n = \frac{2}{\pi}\int_0^{\pi} 4\,Sen(4x)Cos(2x)\,Sen\,(n\pi)dx$$

Resolver aquella integral resulta complicado, pero vamos a realizar un artificio

Sabemos que:

$$Sen(i\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i} \qquad ; \qquad Cos(i\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2}$$

Por lo tanto:

$$4\,Sen(4x)Cos(2x) = 4\left(\frac{e^{i4x} - e^{-i4x}}{2i}\right)\left(\frac{e^{i2x} - e^{-i2x}}{2}\right)$$

$$4\,Sen(4x)Cos(2x) = 2\left(\frac{e^{i6x} + e^{i2x} - e^{-i2x} - e^{-i6x}}{2i}\right)$$

$$4\,Sen(4x)Cos(2x) = 2\left(\frac{e^{i6x} - e^{-i6x}}{2i} + \frac{e^{i2x} - e^{-i2x}}{2i}\right)$$

$$4\,Sen(4x)Cos(2x) = 2\,Sen(6x) + 2\,Sen(2x)$$

Además sabemos que:

$$f(x) = \sum_{n=1}^{\infty} C_n\,Sen(nx)$$

$$2\,Sen(6x) + 2\,Sen(2x) = C_1Sen(x) + C_2Sen(2x) + C_3Sen(3x) + C_4Sen(4x) + C_5Sen(5x) + C_6Sen(6x) + \;\ldots\ldots\ldots$$

Entonces:

$$C_2 = 2 \quad ; \quad C_6 = 2 \quad ; \quad C_n = 0 \quad , \quad \forall n \in N - \{2,6\}$$

Finalmente la ecuación de calor es:

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-(n)^2 t}\,Sen(nx)$$

$$\boxed{\boldsymbol{u(x,t) = 2e^{-4t}Sen(2x) + 2e^{-36t}Sen(6x)}}$$

3) Determine la solución:

$$\boldsymbol{2\,u_t = u_{xx} \quad ; \quad 0 < x < 1 \quad ; \quad t > 0}$$

$$\boldsymbol{u(0,t) = u(1,t) = 0}$$

$$\boldsymbol{u(x,0) = 4\,Sen(\pi x)Cos^3(\pi x)}$$

Determinando C_n

$$C_n = 2\int_0^1 4\,Sen(\pi x)Cos^3(\pi x)\,Sen\,(n\pi)dx$$

Resulta complicado resolver la integral, entonces:

$$4\,Sen(\pi x)Cos^3(\pi x) = 4\left(\frac{e^{i\pi x} - e^{-i\pi x}}{2i}\right)\left(\frac{e^{i\pi x} - e^{-i\pi x}}{2}\right)^3$$

$$4\,Sen(\pi x)Cos^3(\pi x) = \frac{1}{2}\left[\frac{\left(e^{i\pi x} - e^{-i\pi x}\right)\left(e^{i3\pi x} + 3e^{i\pi x} + 3e^{-i\pi x} + e^{-i3\pi x}\right)}{2i}\right]$$

$$4\,Sen(\pi x)Cos^3(\pi x) = \frac{1}{2}\left(\frac{e^{i4\pi x} + 3e^{i2\pi x} + 3e^{0} + e^{-i2\pi x} - e^{-i2\pi x} - 3e^{0} - 3e^{-i2\pi x} - e^{-i4\pi x}}{2i}\right)$$

$$4\,Sen(\pi x)Cos^3(\pi x) = \frac{1}{2}\left(\frac{e^{i4\pi x} - e^{-i4\pi x}}{2i} + 2\frac{e^{i2\pi x} - e^{-i2\pi x}}{2i}\right)$$

$$4\,Sen(\pi x)Cos^3(\pi x) = \frac{1}{2}\left[Sen(4\pi x) + 2\,Sen(2\pi x)\right]$$

Ahora:

$$Sen(2\pi x) + \frac{1}{2}Sen(4\pi x) = C_1 Sen(\pi x) + C_2 Sen(2\pi x) + C_3 Sen(3\pi x) + C_4 Sen(4\pi x) + C_5 Sen(5\pi x) + \ldots\ldots$$

Entonces:

$$C_2 = 1 \quad ; \quad C_4 = \frac{1}{2} \quad ; \quad C_n = 0 \quad , \quad \forall n \,\epsilon\, N - \{2,4\}$$

Finalmente la ecuación de calor es:

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\left(\frac{n\pi}{\sqrt{2}}\right)^2 t}\,Sen(n\pi x)$$

$$u(x,t) = \sum_{n=1}^{\infty} C_n\, e^{-\frac{(n\pi)^2}{2} t}\,Sen(n\pi x)$$

$$\boxed{\boldsymbol{u(x,t) = e^{-2\pi^2 t} Sen(2\pi x) + \frac{1}{2} e^{-8\pi^2 t} Sen(4\pi x)}}$$

ECUACIÓN DE LA ONDA

La ecuación de la onda es:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Resolución de la ecuación de la onda

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \, Cos\left(\frac{n\pi c}{l} t\right) + B_n \, Sen\left(\frac{n\pi c}{l} t\right) \right] Sen\left(\frac{n\pi}{l} x\right)$$

Encontrando A_n

$$A_n = \frac{2}{l} \int_0^l f(x) \, Sen\left(\frac{n\pi}{l} x\right) dx$$

Donde f(x) es el desplazamiento de la cuerda

$$f(x) = \sum_{n=1}^{\infty} A_n \, Sen\left(\frac{n\pi}{l} x\right)$$

Encontrando B_n

$$B_n = \frac{2}{n\pi c} \int_0^l g(x) \, Sen\left(\frac{n\pi}{l} x\right) dx$$

Donde g(x) es la velocidad de la cuerda

$$g(x) = \sum_{n=1}^{\infty} \left(\frac{n\pi c}{l} B_n\right) Sen\left(\frac{n\pi}{l} x\right)$$

1) La figura muestra la función de la posición inicial f(x) de una cuerda de la longitud L = 2m estirada, que es puesta en movimiento al colocar su punto medio x = L/2 = 1m a una distancia de 2m desde la línea de referencia horizontal y soltándola de esa posición de reposo a partir del tiempo t = 0. Considere que la constante $c^2 = 4$

Pero f(x) presenta dos tramos, por lo que vamos a hallar las correspondientes reglas de correspondencia

Ecuación de la recta conociendo 2 puntos:

$$\frac{y-y_2}{x-x_2}=\frac{y_2-y_1}{x_2-x_1}$$

Ecuación de la recta 1: Q(1,2) ; R(2,0) :

$$\frac{y-0}{x-2}=\frac{0-2}{2-1} \quad \Rightarrow \quad y=-2x+4$$

Ecuación de la recta 2: P(0,0) ; Q(1,2) :

$$\frac{y-2}{x-1}=\frac{2-0}{1-0} \quad \Rightarrow \quad y=2x$$

Entonces:

$$f(x)=\begin{cases} 2x & ;\ 0\le x\le 1 \\ -2x+4 & ;\ 1<x\le 2 \end{cases}$$

Encontrando A_n

$$A_n=\frac{2}{2}\int_0^2 f(x)\,Sen\left(\frac{n\pi}{2}x\right)dx$$

$$A_n=\int_0^1 2x\,Sen\left(\frac{n\pi}{2}x\right)dx+\int_1^2(-2x+4)\,Sen\left(\frac{n\pi}{2}x\right)dx$$

$$A_n=2\int_0^1 x\,Sen\left(\frac{n\pi}{2}x\right)dx-2\int_1^2 x\,Sen\left(\frac{n\pi}{2}x\right)dx+4\int_1^2 Sen\left(\frac{n\pi}{2}x\right)dx$$

Resolviendo:

$$\int x\,Sen\left(\frac{n\pi}{2}x\right)dx$$

Integrando por partes:

$$u=x \quad \Rightarrow \quad du=dx$$

$$dv=Sen\left(\frac{n\pi}{2}x\right)dx \quad \Rightarrow \quad v=-\frac{2}{n\pi}Cos\left(\frac{n\pi}{2}x\right)$$

Luego:

$$\int x\,Sen\left(\frac{n\pi}{2}x\right)dx = -\frac{2x}{n\pi}Cos\left(\frac{n\pi}{2}x\right) + \frac{2}{n\pi}\int Cos\left(\frac{n\pi}{2}x\right)dx$$

$$\int x\,Sen\left(\frac{n\pi}{2}x\right)dx = -\frac{2x}{n\pi}Cos\left(\frac{n\pi}{2}x\right) + \left(\frac{2}{n\pi}\right)^2 Sen\left(\frac{n\pi}{2}x\right)$$

Por lo tanto:

$$A_n = 2\left[-\frac{2x}{n\pi}Cos\left(\frac{n\pi}{2}x\right) + \left(\frac{2}{n\pi}\right)^2 Sen\left(\frac{n\pi}{2}x\right)\right]_0^1 - 2\left[-\frac{2x}{n\pi}Cos\left(\frac{n\pi}{2}x\right) + \left(\frac{2}{n\pi}\right)^2 Sen\left(\frac{n\pi}{2}x\right)\right]_1^2 - \frac{8}{n\pi}\left[Cos\left(\frac{n\pi}{2}x\right)\right]_1^2$$

$$A_n = 2\left[-\frac{2}{n\pi}Cos\left(\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\right)^2 Sen\left(\frac{n\pi}{2}\right)\right] - 2\left[-\frac{4}{n\pi}Cos(n\pi) + \left(\frac{2}{n\pi}\right)^2 Sen(n\pi) + \frac{2}{n\pi}Cos\left(\frac{n\pi}{2}\right) - \left(\frac{2}{n\pi}\right)^2 Sen\left(\frac{n\pi}{2}\right)\right] - \frac{8}{n\pi}\left[Cos(n\pi) - Cos\left(\frac{n\pi}{2}\right)\right]$$

$$A_n = 2\left[-\frac{2}{n\pi}Cos\left(\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\right)^2 Sen\left(\frac{n\pi}{2}\right)\right] - 2\left[-\frac{4}{n\pi}Cos(n\pi) + \frac{2}{n\pi}Cos\left(\frac{n\pi}{2}\right) - \left(\frac{2}{n\pi}\right)^2 Sen\left(\frac{n\pi}{2}\right)\right] - \frac{8}{n\pi}\left[Cos(n\pi) - Cos\left(\frac{n\pi}{2}\right)\right]$$

$$A_n = -\frac{4}{n\pi}Cos\left(\frac{n\pi}{2}\right) + 2\left(\frac{2}{n\pi}\right)^2 Sen\left(\frac{n\pi}{2}\right) + \frac{8}{n\pi}Cos(n\pi) - \frac{4}{n\pi}Cos\left(\frac{n\pi}{2}\right) + 2\left(\frac{2}{n\pi}\right)^2 Sen\left(\frac{n\pi}{2}\right) - \frac{8}{n\pi}Cos(n\pi) + \frac{8}{n\pi}Cos\left(\frac{n\pi}{2}\right)$$

$$A_n = -\frac{8}{n\pi}Cos\left(\frac{n\pi}{2}\right) + \frac{8}{n\pi}Cos\left(\frac{n\pi}{2}\right) + \frac{8}{n\pi}Cos(n\pi) - \frac{8}{n\pi}Cos(n\pi) + 4\left(\frac{2}{n\pi}\right)^2 Sen\left(\frac{n\pi}{2}\right)$$

$$A_n = \frac{16}{(n\pi)^2}Sen\left(\frac{n\pi}{2}\right)$$

Cuando n es par tenemos que:

$$A_{2n} = \frac{16}{(2n\pi)^2}Sen\left(\frac{2n\pi}{2}\right)$$

$$A_{2n} = \frac{16}{(2n\pi)^2}Sen(n\pi) \quad \Rightarrow \quad A_{2n} = 0$$

Cuando n es impar tenemos que:

$$A_{2n-1} = \frac{16}{[(2n-1)\pi]^2}Sen\left[\frac{(2n-1)\pi}{2}\right]$$

$$A_{2n-1} = \frac{16}{[(2n-1)\pi]^2}Sen\left(n\pi - \frac{\pi}{2}\right)$$

Si resolvemos:

$$Sen\left(n\pi - \frac{\pi}{2}\right) = Sen(n\pi)\,Cos\left(\frac{\pi}{2}\right) - Cos(n\pi)\,Sen\left(\frac{\pi}{2}\right)$$

$$Sen\left(n\pi - \frac{\pi}{2}\right) = -Cos(n\pi)$$

Entonces:

$$A_{2n-1} = \frac{16}{[(2n-1)\pi]^2}[-Cos(n\pi)]$$

$$A_{2n-1} = \frac{16}{[(2n-1)\pi]^2}[-(-1)^n]$$

$$A_{2n-1} = \frac{16}{[(2n-1)\pi]^2}(-1)^{n+1}$$

Ojo $B_n = 0$ *, debido a que la cuerda parte del reposo, es decir g(x)=0*

Entonces la solución de la ecuación de la onda descrita por la cuerda está dada por:

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n\, Cos\left(\frac{n\pi(2)}{2}t\right)\right] Sen\left(\frac{n\pi}{2}x\right)$$

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_{2n-1}\, Cos[(2n-1)\pi\, t]\right] Sen\left[\frac{(2n-1)\pi}{2}x\right]$$

$$\boxed{\boldsymbol{u(x,t) = \sum_{n=1}^{\infty}\frac{16(-1)^{n+1}}{[(2n-1)\pi]^2} Sen\left[\frac{(2n-1)\pi}{2}x\right] Cos[(2n-1)\pi\, t]}}$$